Mechanical Vibrations

(R18A2117)

COURSE FILE

IV B. Tech I Semester

(2022-2023)

Prepared By

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&

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Department of Aeronautical Engineering

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. ofIndia)

Affiliated to JNTU, Hyderabad, Approved by AICTE - Accredited by NBA & NAAC – 'A' Grade - ISO 9001:2015 Certified)

Maisammaguda, Dhulapally (Post Via. Kompally), Secunderabad – 500100, Telangana State, India.

MRCET VISION

To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become pioneers of Indian vision of modern society.

MRCET MISSION

- To become a model institution in the fields of Engineering, Technology and Management.
- To have a perfect synchronization of the ideologies of MRCET with challenging demands of International Pioneering Organizations.

MRCET QUALITY POLICY.

- To pursue continual improvement of teaching learning process of Undergraduate and Post Graduate programs in Engineering & Management vigorously.
- To provide state of art infrastructure and expertise to impart the quality education.

PROGRAM OUTCOMES
(PO's)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design / development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multi disciplinary environments.
12. **Life- long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

DEPARTMENT OF AERONAUTICAL ENGINEERING

VISION

Department of Aeronautical Engineering aims to be indispensable source in Aeronautical Engineering which has a zeal to provide the value driven platform for the students to acquire knowledge and empower themselves to shoulder higher responsibility in building a strong nation.

MISSION

The primary mission of the department is to promote engineering education and research. To strive consistently to provide quality education, keeping in pace with time and technology. Department passions to integrate the intellectual, spiritual, ethical and social development of the students for shaping them into dynamic engineers.

QUALITY POLICY STATEMENT

Impart up-to-date knowledge to the students in Aeronautical area to make them quality engineers. Make the students experience the applications on quality equipment and tools. Provide systems, resources and training opportunities to achieve continuous improvement. Maintain global standards in education, training and services.

PROGRAM EDUCATIONAL OBJECTIVES – Aeronautical Engineering

1. **PEO1 (PROFESSIONALISM & CITIZENSHIP):** To create and sustain a community of learning in which students acquire knowledge and learn to apply it professionally with due consideration for ethical, ecological and economic issues.
2. **PEO2 (TECHNICAL ACCOMPLISHMENTS):** To provide knowledge based services to satisfy the needs of society and the industry by providing hands on experience in various technologies in core field.
3. **PEO3 (INVENTION, INNOVATION AND CREATIVITY):** To make the students to design, experiment, analyze, and interpret in the core field with the help of other multi disciplinary concepts wherever applicable.
4. **PEO4 (PROFESSIONAL DEVELOPMENT):** To educate the students to disseminate research findings with good soft skills and become a successful entrepreneur.
5. **PEO5 (HUMAN RESOURCE DEVELOPMENT):** To graduate the students in building national capabilities in technology, education and research

PROGRAM SPECIFIC OUTCOMES – Aeronautical Engineering

1. To mould students to become a professional with all necessary skills, personality and sound knowledge in basic and advance technological areas.
2. To promote understanding of concepts and develop ability in design manufacture and maintenance of aircraft, aerospace vehicles and associated equipment and develop application capability of the concepts sciences to engineering design and processes.
3. Understanding the current scenario in the field of aeronautics and acquire ability to apply knowledge of engineering, science and mathematics to design and conduct experiments in the field of Aeronautical Engineering.
4. To develop leadership skills in our students necessary to shape the social, intellectual, business and technical worlds.

(R18A2118)Mechanical Vibrations

UNIT–I

FUNDAMENTALS OF VIBRATION: Brief history of vibration, Importance of the study of vibration, basic concepts of vibration, classification of vibrations, vibration analysis procedure, spring elements, mass or inertia elements, damping elements.

FREE VIBRATION OF SINGLE DEGREE OF FREEDOM SYSTEMS: Introduction, Free vibration of an undamped translational system, free vibration of an undamped torsional system. Concepts on different damping conditions.

UNIT-II

HARMONICALLY EXITED VIBRATIONS: Introduction, Equation of motion, response of an undamped system under harmonic force, Response of a damped system under harmonic force, forced vibration with coulomb damping, forced vibration with hysteresis damping.

UNIT-III

VIBRATION UNDER GENERAL FORCING CONDITIONS: Introduction, Response under a general periodic force, Two Degree of Freedom Systems: Introduction, Equation of motion for forced vibration, free vibration analysis of an undamped system, Torsional system, forced vibration analysis.

UNIT–IV

MULTIDEGREE OF FREEDOM SYSTEMS: Introduction, Modeling of Continuous systems as multi degree of freedom systems, Using Newton's second law to derive equations of motion, Determination Of Natural Frequencies and Mode Shapes: Introduction, Dunkerley's formula, Rayleigh's method, Holzers method, Matrix iteration method, Jacobi's method.

Unit-V

CONTINUOUS SYSTEMS: Transverse vibration of a spring or a cable, longitudinal vibration of bar or rod, Torsional vibration of a bar or rod, Lateral vibration of beams.

SuggestedTextBooks:

1: Mechanical Vibrations by S. S. Rao
2: Mechanical Vibrations by V. P. Singh
3: Mechanical Vibrations by G. K. Grover

CourseOutcomes:

Fundamental frequency of Multi- DOF systems can estimate by various methods.
Effect of unbalance in rotating masses has been studied.
How to determine eigenvalues and eigenvectors for a vibratory system has analysed

R17

Code No: R17A0368

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

IV B.TechI Semester Regular Examinations, February 2021

Mechanical Vibrations & Structural Dynamics

(AE)

Roll No										

Time: 2 hours 30 min **Max. Marks: 70**

Answer Any **Five** Questions

All Questions carries equal marks.

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1 Write short notes on the following:

(a) Types of vibration with examples **[8M]**

(b) Harmonic analysis **[6M]**

2 (a) Consider a pendulum of length 1 unit and mass m kg, is restricted to one plane only. Derive natural frequency by: (i) Newtons second law of motion (ii) Energy method **[8M]**

(b) Determine the natural frequency of the spring mass system considering the inertial effect of the spring. **[6M]**

3 (a) Derive the frequency response relation for a single degree of freedom system subjected to harmonic excitation. Also plot the frequency response diagram. **[7M]**

(b) The damped natural frequency of a system is 10.5 Hz. When the system is subjected to a harmonic excitation the maximum amplitude of oscillations occurs at 9.5 Hz. Find the amplitude ratio when it is excited at 15 Hz frequency? **[7M]**

4 (a) Derive the frequency response relation for base excitation. **[6M]**

(b) A vehicle of mass 400 kg, total spring constant of its suspension 500 N/cm and the damping factor 0.2 moves over.a road which may be approximatedtoa sine wave of amplitude 3 cm and wavelength 4m. Determine: **[8M]**

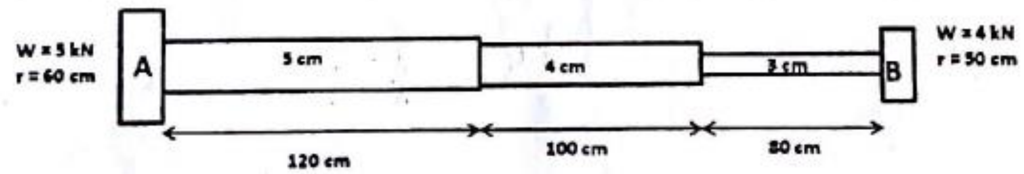
i) the critical speed of the vehicle

ii) the amplitude of the steady state motion at the critical speed and

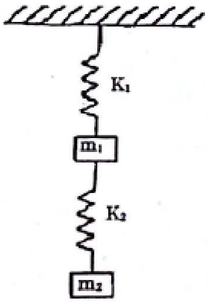
iii) the amplitude of the vehicle when it is driven at a speed 1.5 times more than the' critical speed.

5 (a) Derive the equations of motion for a two-rotor system and frequencyis evaluated from those equations. **[7M]**

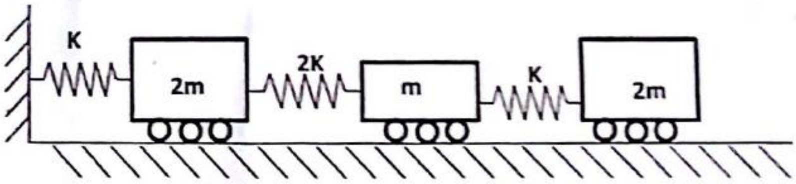
(b) Find the natural frequency of the two-rotor system shown in figure. Also locate the position of the node with respect to therotor A. G = 70 GPa. **[7M]**



6 Determine the equations of motion and the natural frequencies of the two degree freedom spring-mass system shown in figure below. **[14M]**



7 Using matrix iteration find the frequency of the system shown in the figure below:



[14M]

8 Derive expression for governing differential equation for torsional vibration of circular or uniform shafts. [14M]

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R15

Code No: R15A0368

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

IV B.TechI Semester Supplementary Examinations, February 2021

Mechanical Vibrations & Structural Dynamics

(AE)

Roll No										

Time: 2 hours 30 min **Max. Marks: 75**

Answer Any **Five** Questions

All Questions carries equal marks.

1 Determine the natural frequency of system in figure 1 **[15M]**

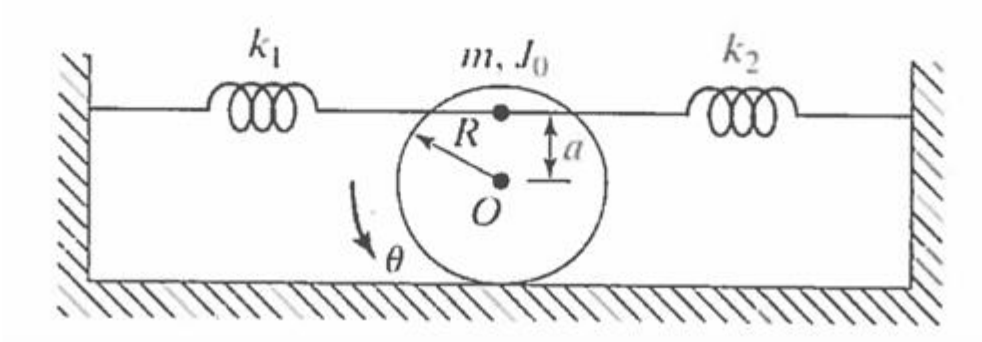

Fig.1

2 body of 5 kg is supported on a spring of stiffness 200 N/m and has dashpot connected to it which produces a resistance of .002 N at the velocity of 1 cm/sec. In what ratio will thee amplitude of vibration to be reduced after 5 cycles. **[15M]**

3 Discuss in detail about vibration measuring instrument Vibrometer and Accelerometer. **[15M]**

4 A machine having a mass of 100 kg and supported on spring of total stiffness 7.84x10^5 N/m has a un unbalanced rotating element which results in disturbing force of 392 N at a speed of 3000 rpm. Assuming a damping factor equals to 0.20. **[15M]**

(a)Determine amplitude of motion due to unbalance,

(b) Transmissibility.

5 Consider a double pendulum of length L_1 =L_2 =L. Determine the natural frequency of system k =100N/m, M_1 =2Kg, M_2= 5 kg L=0.2m, a=0.1m as shown in figure 2. **[15M]**

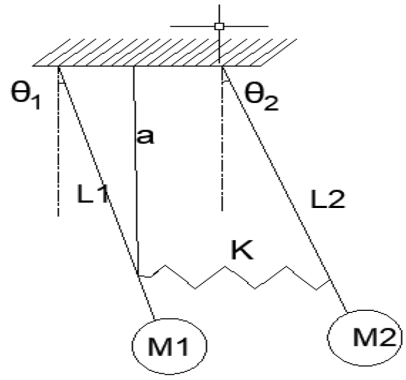

Fig.2

6 Calculate the natural frequency of system of K_1=40N/m , K_2 = 60 N/m M_1= 2Kg , M_2 = 5 kg as shown in figure 3. **[15M]**

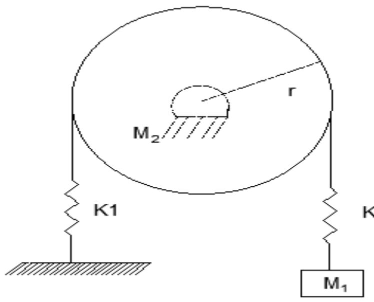

Fig.3

7 Solve for the lowest natural frequency of the system by Rayleigh's method E= 1.96X10^{11} N/m^{2} , I=4X$10^{7}$$m^4$ in figure 4. **[15M]**

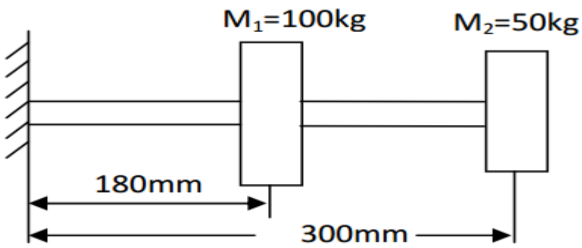

Fig.4

8 Determine the frequency equation for a beam with both ends free having transverse vibrations. **[15M]**

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Code No: **R15A0368**

R15

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

IV B.Tech I Semester Supplementary Examinations, July/August 2021

Mechanical Vibrations & Structural Dynamics

(AE)

Roll No										

Time: 3 hours **Max. Marks: 75**

Answer Any **Five** Questions

All Questions carries equal marks.

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1 Determine the effect of mass of spring on natural frequency of the system from figure1. **[15M]**

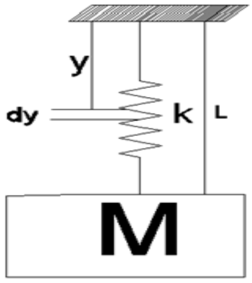

Fig.1

2 A vibratory system is defined by the following parameters m= 3 kg k=100 N/m C= 3Nsec/m **[15M]**

Determine

Damping factor,

The natural frequency of damped vibration

c) Logarithmic decrement

d) Ratio of two consecutive amplitude

The number of cycles after which original amplitude is reduced to 20 percent.

3 Derive the expression of total response of a vibratory system under harmonic force. **[15M]**

4 A vibrating system having total mass 25 kg. A speed of 1000 rpm. The system and eccentric mass have a phase difference of 90 and the corresponding amplitude is 1.5 cm. The eccentric unbalanced mass of 1 kg has the radius of rotation 4 cm. Determine(a) the natural frequency of the system (b) the damping factor (c) the amplitude at 1500 rpm (d)the phase angle at1500 rpm **[15M]**

5 An Aerofoil wing in its first bending and torsional mode is shown in figure-2, Evaluate the equation of motion and obtain its natural frequency if M =5kg, I =0.12 kgm^2 , K =5 x 10^3 N/m , K_T = 0.4 x 10^3 Nm/rad, a=0.1m. **[15M]**

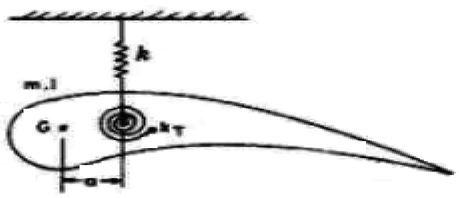

Fig.2

6 Calculate the natural frequency of system if K_1 = 40 =K_3 and K_2 =60, m_1 = m_2=10Kg as shown in figure 3 **[15M]**

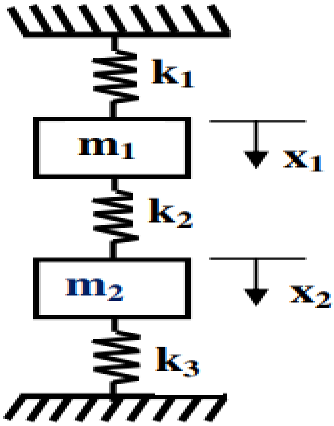

Fig.3

7 Determine the frequencies of system as shown in figure-4 by matrix method if $M_1 = M_2 = M_3 = 10$ kg $K_1 = K_2 = K_3 = 5$ N/m **[15M]**

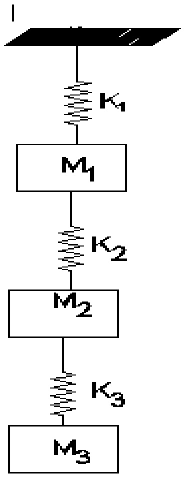

Fig.4

8 Determine the frequency equation of torsional vibrations for a free-free shaft of length L. **[15M]**

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MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

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IV B.Tech I Semester Supplementary Examinations, October 2020

Mechanical Vibrations & Structural Dynamics

(AE)

Roll No										

Time: 2 hours **Max. Marks: 75**

Answer Any **Four** Questions

All Questions carries equal marks.

1 Figure 1 shows a three-stepped shaft fixed at one end and subjected to a torsional moment T at the other end. The length of step i is and its diameter is D_i, i = 1, 2, 3.All the steps aremade of the same material with shear modulus $G_i= G$, i = 1, 2, 3.Find the torsional spring constant (or stiffness) k_{ti} of step i (i = 1, 2, 3). Also find the equivalent torsional spring constant (or stiffness) of the stepped shaft, k_{teq}, so that$T = k_{teq}\theta$.

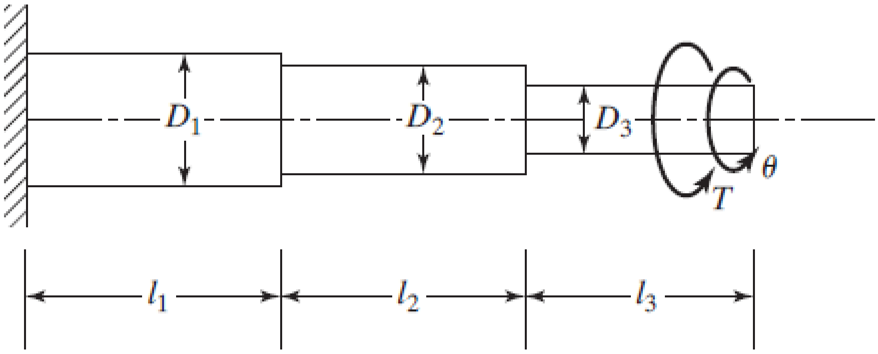

Figure 1

2 The maximum velocity attained by the mass of a simple harmonic oscillator is 10 cm/s, andthe period of oscillation is 2 s. If the mass is released with an initial displacement of 2 cm,find (a) the amplitude, (b) the initial velocity, (c) the maximum acceleration, and (d) thephase angle.

3 a) Find the frequency and amplitude of a single DOF system.
b) A 40 kg mass hangs from a spring with a stiffness of 4 X 10^4 N/m. A harmonicforce with a magnitude of 120 rad/s is applied. Determine the amplitude of theforced response.

4 a) Explain why vibration isolation is difficult at low speeds.
b) A 5 kg block is mounted on a helical coil spring such that the system's natural frequency is 50 rad/s. The block is subject to a harmonic excitation of amplitude 45 N at a frequency of 50.8 rad/s. What is the maximum displacement of the block from its equilibrium positions?

5 Use the Duhamel's integral method to derive expressions for the response of an undamped system subjected to the forcing function shown in figure 2.

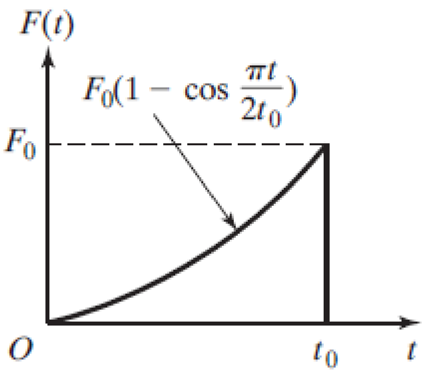

Figure 2

6 a) How are the initial conditions determined for a single-degree-of-freedom system subjected to an impulse at $t = 0$?
b) During operation, a 100 kg reciprocating machine is subject to a force F(t) = 200 sin 60t

N. The machine is mounted on springs of an equivalent stiffness of 4.3 X 106 N/m. What is the machine’s steady-state amplitude?

7 Find the flexibility and stiffness influence coefficients of the system shown in figure 3. Also, derive the equations of motion of the system.

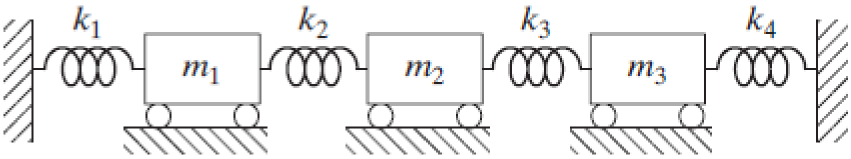

Figure 3

8 A uniform bar of cross-sectional area *A*, length *l*, and Young s modulus *E* is connected at both endsby springs, dampers, and masses, as shown in figure 4. State the boundary conditions.

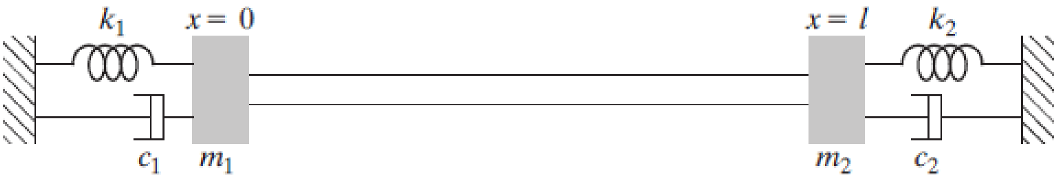

Figure 4

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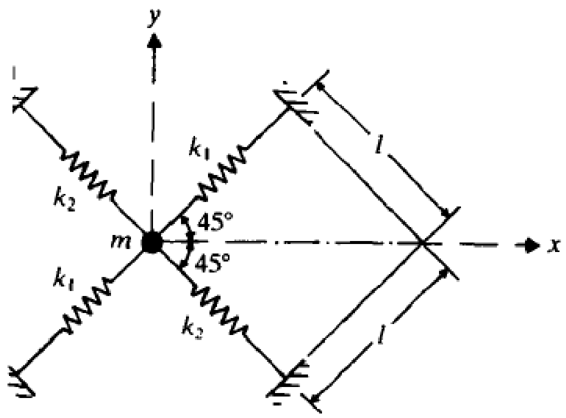
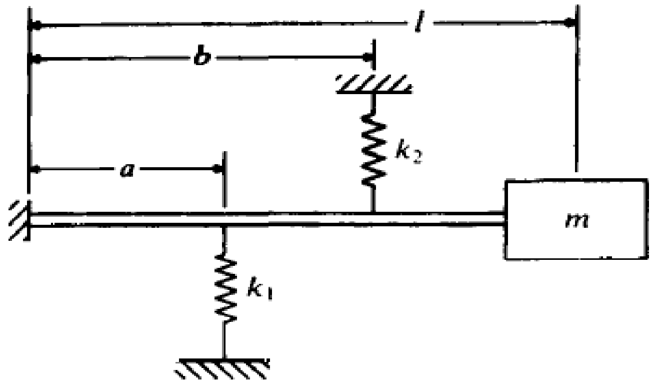
M V Important Questions

1] Figure shows a small mass ‘m’ restrained by four linearly elastic springs, each of which has an unstretched length l, and an angle of orientation of 45° with respect to the x-axis.Determine the equation of motion for small displacements of the mass in the x-direction.



2] Find the natural frequency of the system shown in figure with and without the springs k_1 and k_2 in the middle of the elastic beam.



3] A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the frequency is altered by 45%. Find the mass and spring constant of the original system.

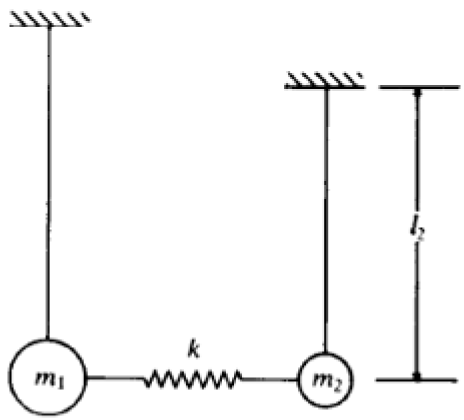
4] Describe the basic concepts of vibration, classification of vibrations, and vibration analysis procedure.

5] Explain the Response of an Undamped System Under Harmonic Force.

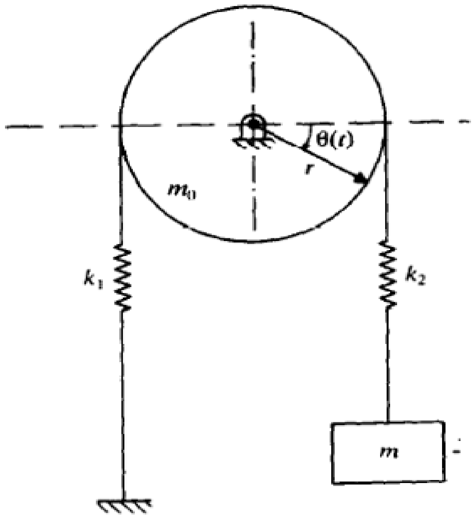
6] A mass ‘m’ is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force having amplitude of 100 N and a frequency of 5 Hz. The amplitude of the forced motion of the mass is observed to be 20 mm. find the value of ‘m’.

7] Describe theForced Vibration with Coulomb Damping.

8] Find the natural frequencies and the amplitude ratios of the system shown in figure.



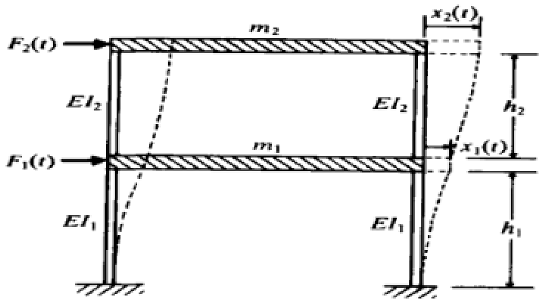
9] Determine the natural frequencies of the system shown in figure by assuming the rope passing over the cylinder does not slip.



10] A two story building frame is modeled as shown in figure. The girders are assumed to be rigid and the columns have flexural rigidities EI_1 and EI_2, with negligible masses. The stiffness of each column can be computed as

$$24EI_i/h_i^3, \quad i = 1, 2$$

For $m_1 = 2m$, $m_2 = m$, $h_1 = h_2 = h$ and $EI_1 = EI_2 = EI$, determine the natural frequencies.



UNIT-1
FUNDAMENTALS OF VIBRATION

Brief history of vibration

The discovery of musical instruments such as drums, whistles etc. made the vibration known and more interesting to the scientists and engineers. It was known since long that sound is related to vibration; but no mathematical relation was available. **Galileo** (1564-1642), an Italian mathematician, studied the oscillations of strings and simple pendulum. He developed mathematical relationship between the length of a pendulum and its frequency and discussed the term resonance. The Galileo and Hooke's developed relationship between the frequency and pitch of sound.

Sir Isaac Newton (1642-1727), an English mathematician, made a lot of scientific contribution towards dynamics by introducing the definition of Forces, Mass, Momentum and three Laws of motion.

Daniel Bernoulli (1700-1782) developed the equation of motion for vibrations of beams and studied the vibrating strings and discovered the principle of superposition of harmonics in free vibration.

L. Euler (1707-1783) worked on the bending vibrations of a rod and studied the dynamics of a vibrating ring. **J. B. J. Fourier** (1768-1830) was a French mathematician who made valuable contribution to the development of vibration theory. He has shown that any periodic function can be represented by a series of sines and cosines. This work of Fourier helps in analysing the experimentally obtained vibration plots analytically. Lord Rayleigh (1842-1919), an English physicist, hascomputed the approximately natural frequencies of vibrating bodies using an energy approach. The method derived by him is useful in developing the equations of motion and the technique is known as Rayleigh's method.

A lot of work has been done in vibration by many authors. About thirty years back, the vibration analysis of complex multi-degree of freedom systems was very difficult. But now with the help of finite element method and other advanced techniques the engineers are able to use computers to conduct numerically detailed vibration analysis of complex mechanical systems even having thousands degree of freedom.

Importance of the study of vibration

Most human activities involve vibration in one form or other. Example, we hear because our eardrums vibrate and see because light waves undergo vibration. Breathing is associated with the vibration of lungs and walking involves (periodic) oscillatory motion of legs and hands. We speak due to the oscillatory motion of larynges (tongue).

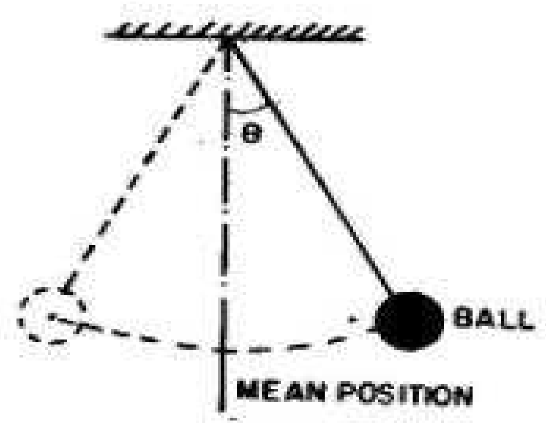
Similarly, the structures designed to support the high speed engines and turbines are subjected to vibration. Due to faulty design and poor manufacture there is unbalance in the engines which causes excessive and unpleasant stresses in the rotating system because of vibration. The vibration causes rapid wear of machine parts such as bearings and gears. Unwanted vibrations may cause loosening of parts from the machine. Because of improper design or material distribution, the wheels of locomotive can leave the track due to excessive vibration which results in accident or heavy loss. As we know that many buildings, structures and bridges fall because of vibration. If the frequency of excitation coincides with one of thenatural frequencies of the system, a condition of resonance (by the synchronous vibration of a neighbouring object)is reached, and dangerously large oscillations may occur which may result in the mechanical failure of the system. Excessive vibration is dangerous for human beings. Thus keeping in view all these devasting effects, the study of vibration is essential for a Mechanical/Aeronautical/Design engineers to minimize the vibrational effects over mechanical components by designing them suitably.

Thus, undesirable vibrations should be eliminated or reduced upto certain extent by the following methods:
1] Removing external excitation, if possible.
2] Using Shock absorbers.
3] Dynamic Absorbers.
4] Resting the system on proper vibration isolators.

Basic Concepts of Vibration

With the discovery of musical instruments like drums, the vibration became a point of interest for scientists and since then there has been much investigation in the field of vibration. All bodies having mass and elasticity are capable of vibration. The mass is inherent of the body and elasticity causes relative motion among its parts. When body particles are displaced by the application of external force, the internal forces in the form of elastic energy are present in the body. These forces try to bring the body to its original position. At equilibrium position, the whole of the elastic energy is converted into kinetic energy and body continues to move in the opposite direction because of it. The whole of the kinetic energy is again converted into elastic or strain energy due to which the body again returns to the equilibrium position. In this way, vibratory motion is repeated indefinitely and exchange of energy takes place. Thus, any motion which repeats itself after an interval of time is called vibration or oscillation.

The swinging of simple pendulum as shown in fig. 1 is an example of vibration or oscillation as the motion of ball is to and fro from its mean position repeatedly.
The main reasons of vibration are as follows:
1. Unbalanced centrifugal force in the system. This is caused because of non-uniform material distribution in a rotating machine element.
2. Elastic nature of the system.
3. External excitation applied on the system.
4. Winds may cause vibrations of certain systems such as electricity lines, telephone lines, etc.



Classification of Vibrations

Vibrations can be classified in several ways. Some of the important classifications are as follows:

Free Vibration: If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as free vibration. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

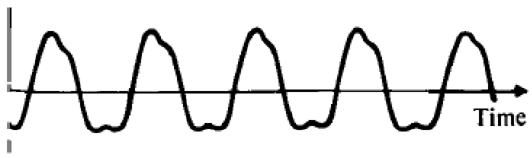
Forced Vibration: If a system is subjected to an external force, the resulting vibration is known as forced vibration. The oscillation that arises in machines such as diesel engines is an example of forced vibration.

If the frequency of the external force coincides with one of the natural frequencies of the system, this condition is known as resonance and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines and airplane wings have been associated with the occurrence of resonance.

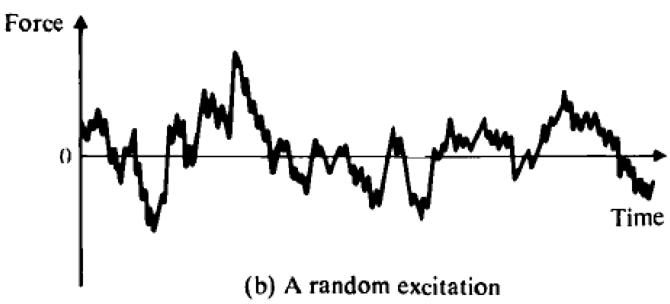
If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as undamped vibration. If any energy is lost in this way, on the other hand, it is called damped vibration. In many physical systems, the amount of damping is so small that it can be disregarded for most engineering purposes. However, consideration of damping becomes extremely important in analyzing vibratory systems near resonance.

If all the basic components of a vibratory system – the spring, the mass and the damper – behave linearly, the resulting vibration is known as linear vibration, on the other hand, if any of the basic components behave non linearly, the vibration is called non linear vibration. The differential equations that govern the behaviour of linear and non linear vibratory systems are linear and non linear respectively. If the vibration is linear, the principle of superposition holds, and the mathematical techniques of analysis are well developed. For non linear vibration, the superposition principle is not valid, and techniques of analysis are less well known. Since all vibratory systems tend to behave non linearly with increasing amplitude of oscillation, a knowledge of non linearly vibration is desirable in dealing with practical vibratory systems.

If the value of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called deterministic. The resulting vibration is known as deterministic vibration. In some cases, the excitation is non deterministic or random; the value of the excitation at a given time cannot be predicted.



(a) A deterministic (periodic) excitation



(b) A random excitation

Vibration Analysis Procedure

A vibratory system is a dynamic system for which the variables such as the excitations (inputs) and responses (output) are time-dependent. The response of vibrating system generally depends on the initial conditions as well as the external excitations. The analysis of a vibrating system usually involves mathematical modelling, derivation of the governing equations, solution of the equations, and interpretation of the results.

Step 1] Mathematical Modelling: The purpose of mathematical modelling is to represent all the important features of the system for the purpose of deriving the mathematical (or analytical) equations governing the behaviour of the system. The mathematical model should include enough details to be able to describe the system in terms of equations without making it too complex. The mathematical model may be linear or non linear depending on the behaviour of the components of the systems. Sometimes the mathematical model is gradually improved to obtain most accurate results.

Step 2] Derivation of Governing Equations: Once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system. The equations are usually in the form of a set of ordinary differential equations for a discrete system and partial differential equations for a continuous system. The equations may be linear or non linear depending on the behaviour of the components of the system. Several approaches are commonly used to derive the governing equations. Among them are Newton's second law of motion, d'Alembert principle and the principle of conservation of energy.

Step 3] Solution of the Governing Equations: The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem, we can use one of the following techniques for finding the solution: standard methods of solving differential equations, Laplace transformation methods, matrix methods, and numerical methods. If the governing equations are non linear, they can seldom be solved in closed form. Further, the solution of partial differential equations is far more involved than that of ordinary differential equations. Numerical methods, using computers, can be used to solve the equations. However, it will be difficult to draw general conclusions about the behaviour of the system using computer results.

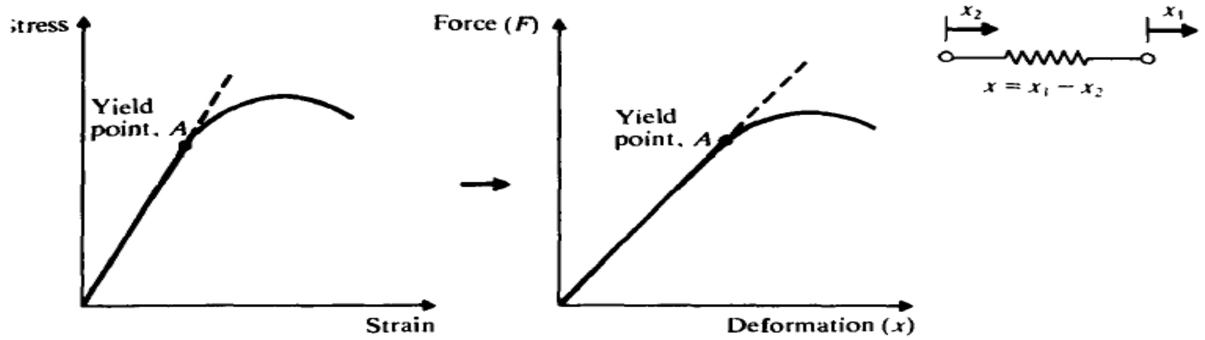
Step 4] Interpretation of the Results: The solution of the governing equations gives the displacements, velocities and accelerations of the various masses of the system. These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

Spring Elements

A linear spring is a type of mechanical link which is generally assumed to have negligible mass and damping. A force is developed in the spring whenever there is relative motion between the two ends of the spring. The spring force is proportional to the amount of deformation and is given by

$$F = -kx \text{ -------- } (1)$$

Where F is the *spring force*, x is the *deformation* (displacement of one end with respect to the other), and k is the *spring stiffness* or *spring constant*. The work done in deforming a spring is stored as strain or potential energy in the spring.

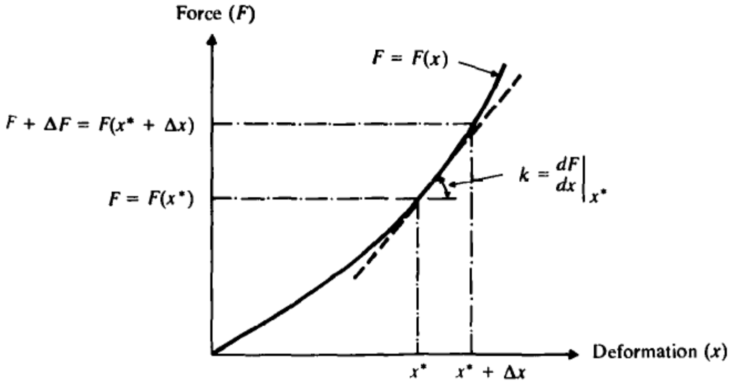


Non linearity beyond proportionality limit

Actual springs are nonlinear and follow equation 1 only up to certain deformation. Beyond a certain value of deformation (after point A in fig.), the stress exceeds the yield point of the material and the force-deformation relation becomes non linear. In many practical applications we assume the deflections to be small and make use of the linear relation in eq. 1.

Even if the force-deflection relation of a spring is non linear, as shown in fig., we often approximate it as a linear one by using a linearization process. To illustrate the linearization process, let the static equilibrium load F acting on the spring cause a deflection of x^*. If an instrumental force ΔF is added to F, the spring deflects by an addition a quantity $\Delta x.$ $the\ new\ spring\ force\ F + \Delta F$ can be expressed using Taylor's series expansion about the static equilibrium position x^* as

$F + \Delta F$ = F(x^* + Δx) = F(x^*) + $\frac{dF}{dx}\Big|(\Delta x) + \frac{1}{2!}\frac{d^2F}{dx^2}\Big|_{x^*}(\Delta x)^2 +$ ---------(2)



Linearization Process

For small values of Δx, the higher order derivative terms can be neglected to obtain

$F + \Delta F$ = F(x^* + Δx) = F(x^*) + $\frac{dF}{dx}|_{x^*}(\Delta$x) ---------(3)

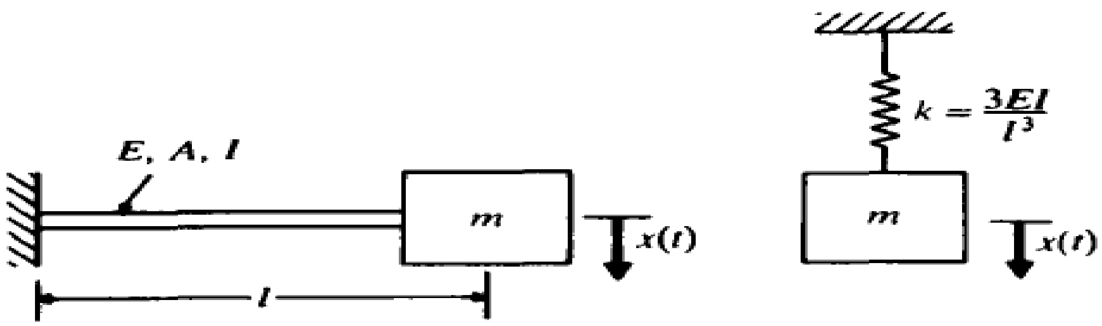
since F = F(x^*), we can express ΔF as

ΔF = kΔx ---------(4)

where k is the linearized spring constant at x^* given by, k = $\frac{dF}{dx}|_{x^*}$

We may use equation 4 for simplicity, but sometimes the error involved in the approximation may be very large.

Elastic elements like beams also behave as springs. For example, consider a cantilever beam with an end mass m, as shown in fig.



(a) Actual system

(b) Single degree of freedom model

We assume, for simplicity, that the mass of the beam is negligible in comparison with the mass m from strength of materials, we know that the static deflection of the beam at the free end is given by, $\delta_{st} = \frac{Wl^3}{3EI}$ --------(5)

where W = mg, is the weight of the mass m, E is the Young's modulus and, I is the moment of inertia of the cross section of the beam. Hence the spring constant is

$$k = \frac{W}{\delta_{st}} = \frac{3EI}{l^3} \text{------- (6)}$$

Similar results can be obtained for beams with different end conditions.

In many practical applications, several linear springs are used in combination, either in Series or in Parallel indicated below

Case (i): Springs in Series: We consider two springs connected in series, as shown in fig. since both the springs are subjected to the same force W, we have for equilibrium

$$W = k_1\delta_1$$

$$W = k_2\delta_2 \text{ -------(7)}$$

Where δ_1 and δ_2 are the elongations of springs 1 and 2, respectively. As the total elongation is equal to the static deflection δ_{st}.

$$\delta_1 + \delta_2 = \delta_{st} \text{ --------(8)}$$

If k_{eq} denotes the equivalent spring constant, then for the same static deflection.

$$W = k_{eq}\delta_{st} \text{ ---------(9)}$$

Equations 7 and 9 gives $k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_{st}$

$$\text{or } \delta_1 = \frac{k_{eq}\delta_{st}}{k_1} \text{ and } \delta_2 = \frac{k_{eq}\delta_{st}}{k_2} \text{ --------(10)}$$

Substituting these values of δ_1 and δ_2 into 8, we obtain

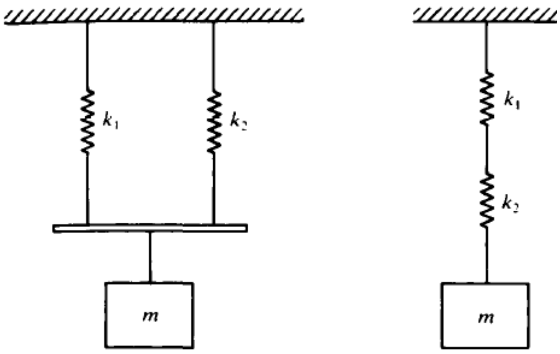
$$\frac{k_{eq}\delta_{st}}{k_1} + \frac{k_{eq}\delta_{st}}{k_2} = \delta_{st}$$

$$\text{i.e. } \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \text{ ------- (11)}$$

Equations 11 can be generalized to the case of n springs in series:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \text{ -------- } + \frac{1}{k_n} \text{ --------(12)}$$

Case (ii): Springs in Parallel. Let the springs be parallel as shown in fig. if W is the weight of mass m, we have for equilibrium, W = $k_1\delta_{st}$ + $k_2\delta_{st}$-------(13)



where $\delta_{st}$is the static deflection of the mass m. if k_{eq}denotes the equivalent spring constant of the combination of the two springs, then for the same static deflection δ_{st}, we have

$$W = k_{eq}\delta_{st} \text{ -------(14)}$$

Equation 13 and 14 give

$$k_{eq} = k_1 + k_2 \text{-------(15)}$$

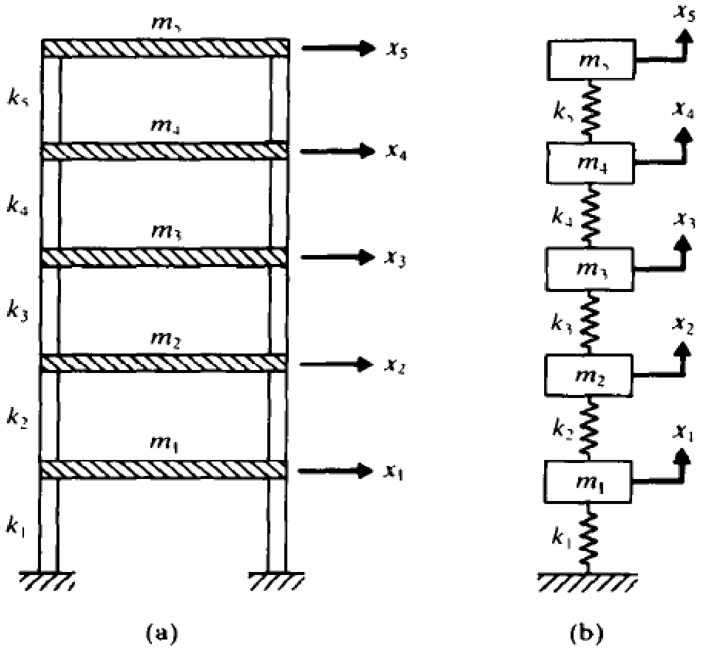
In general, if we have n springs with spring constants k_1, k_2, ……., k_n in parallel, then the equivalent spring constant k_{eq} can be obtained:

$$k_{eq} = k_1 + k_2 + \text{------} + k_n \text{ --------(16)}$$

Mass or Inertia Elements

The mass or inertia element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes. From Newton's second law of motion, the product of the mass and its acceleration is equal to the force applied to the mass. Work is equal to the force multiplied by the displacement in the direction of the force and the work done on a mass is stored in the form of kinetic energy of the mass.

Generally, we uses a mathematical model to represent the actual vibrating system, and there are often several possible models. The purpose of the analysis often determines which mathematical model is appropriate. Once the model is chosen, the mass or inertia elements of the system can be easily identified. For example, consider the cantilever beam with a tip mass shown in fig. For a quick and reasonably accurate analysis, the mass and damping of the beam can be disregarded; the system can be modeled as a spring-mass system as shown in fig. The tip mass m represents the mass element, and the elasticity of the beam denotes the stiffness of the spring. Next, consider a multi-story building subjected to an earthquake. Assuming that the mass of the frame is negligible compared to the masses of the floors, the building can be modeled as a multi-degree of freedom system shown in fig. The masses at the various floor levels represent the mass elements, and the elasticities of the verticalmembers denote the spring elements.

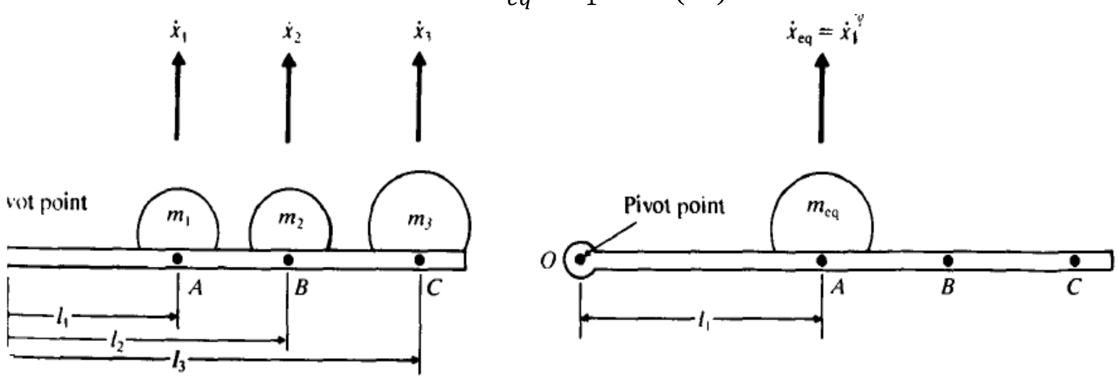


Few Practical Applications:

Case (i): Translational Masses Connected by a Rigid Bar. Let the masses be attached to a rigid bar that is pivoted at one end, as shown in fig. The equivalent mass can be assumed to be located at any point along the bar. To be specific, we assume the location of the equivalent mass to be that of mass m_1**.** The velocities of masses $m_2(\dot{x}_2)$ and $m_3(\dot{x}_3)$ can be expressed in terms of the velocity of mass $m_1(\dot{x}_1)$, by assuming small angular displacements for the bar, as

$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1,\ \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1 \text{------- (17)}$$

$$\text{and} \qquad \dot{x}_{eq} = \dot{x}_1 \text{-------(18)}$$



By equating the kinetic energy of the three mass system to that of the equivalent mass system, we obtain

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 \text{-------}(19)$$

This equation gives, in view of equation 17 and 18,

$$m_{eq} = m_1 + (\frac{l_2}{l_1})^2 m_2 + (\frac{l_3}{l_1})^2 m_3 \text{--------} (20)$$

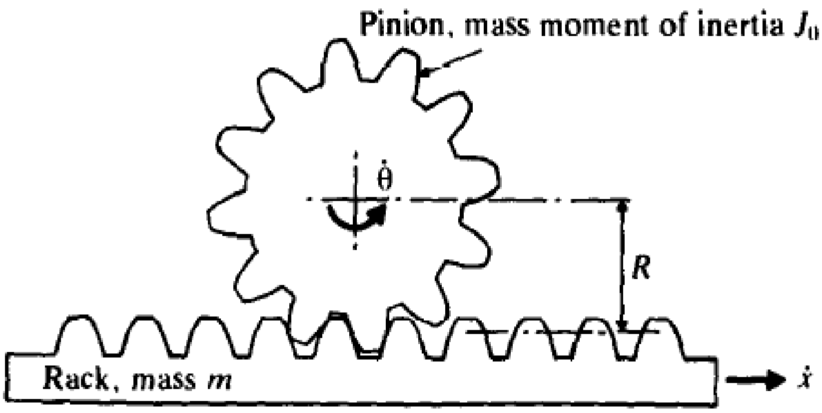
Case (ii): Translational and Rotational Masses Coupled Together. Let a mass m having a traditional velocity $\dot{x}$be coupled to another mass (of mass moment of inertia J_0) having a rotational velocity $\dot{\theta}$, as in the rack and pinion arrangement shown in fig. These two masses can be combined to obtain either 1) a single equivalent translational mass m_{eq} or 2) a single equivalent rotational mass J_{eq}, as shown below.

1. Equivalent translational mass. The kinetic energy of the two masses is given by

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 \text{-------}(21)$$

And the kinetic energy of the equivalent mass can be expressed as

$$T_{eq} = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 \text{--------}(22)$$



Translational and rotational masses in a rack and pinion arrangement

Since $\dot{x}_{eq} = \dot{x}$ and $\dot{\theta} = \frac{\dot{x}}{R}$, the equivalence of T and T_{eq} gives

$$\frac{1}{2}m_{eq}\dot{x}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0(\frac{\dot{x}}{R})^2$$

that is ,

$$m_{eq} = m + \frac{J_0}{R^2} \text{-------} (23)$$

2. Equivalent rotational mass. Here $\dot{\theta}_{eq} = \dot{\theta}$ *and* $\dot{x} = \dot{\theta}R$, and the equivalence of T and T_{eq} leads to

$$\frac{1}{2}J_{eq}\dot{\theta}^2 = \frac{1}{2}m(\dot{\theta}R)^2 + \frac{1}{2}J_0\dot{\theta}^2$$

$$J_{eq} = J_0 + mR^2 \text{-------}(24)$$

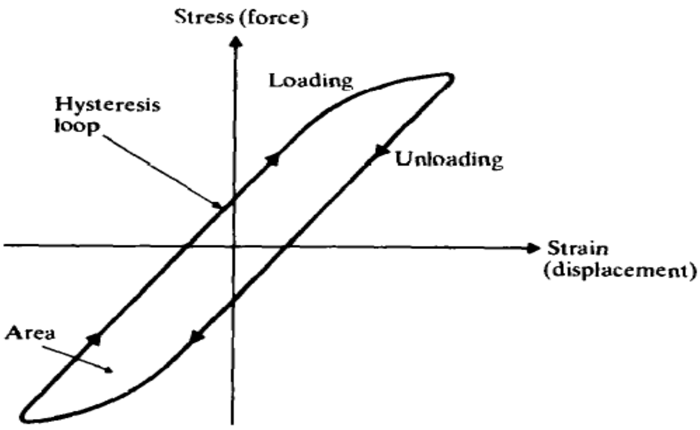
Damping Elements

In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat or sound is known as **Damping**. Although the amount of energy converted into heat or sound is relatively small, the consideration of damping becomes important for an accurate prediction of the vibration response of a system. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. It is difficult to determine the causes of damping in practical systems. Hence damping is modeled as one or more of the following types.

Viscous Damping: It is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, and oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body. Typical examples of viscous damping include (1) fluid film between sliding surfaces. (2) fluid flow around a piston in a cylinder. (3) fluid flow through an orifice, and (4) fluid film around a journal in a bearing.

Coulomb or Dry Friction Damping: Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused due to friction between rubbing surfaces that are either dry or have insufficient lubrication.

Material or Solid or Hysteretic Damping: When materials are deformed, energy is absorbed and dissipated by the material. The effect is due to friction between the internal planes, which slip or slide as the deformations take place. When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as shown in fig. the area of this loop dented the energy lost per cycle due to damping.



Hysteresis loop for elastic material

Definitions

Periodic Motion: A motion which repeats itself after equal intervals of time.

Time Period: Time taken to complete one cycle.

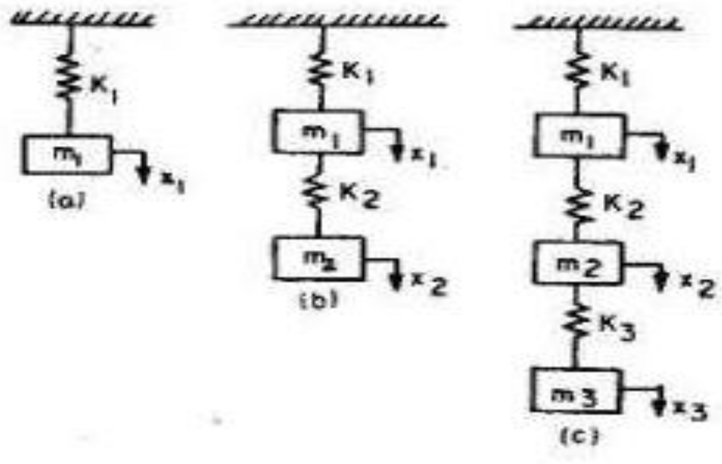
Frequency: Number of cycles per unit time.

Amplitude: The maximum displacement of a vibrating body from its equilibrium position.

Natural Frequency: when no external force acts on the system after giving it an initial displacement, the body vibrates. These vibrations are called free vibrations and their frequency as natural frequency. It is expressed in rad/sec or Hertz.

Fundamental Mode of Vibration: The fundamental mode of vibration of a system in the mode having the lowest natural frequency.

Degree of Freedom: The minimum number of independent coordinates required to specify the motion of a system at any instant is known as degrees of freedom of the system.In general, it is equal to the number of independent displacements that are possible. The one, two and three degrees of freedom systems are shown in figure.



Simple Harmonic Motion: The motion of a body to and fro about a fixed point is called simple harmonic motion. The motion is periodic, and its acceleration is always directed towards the mean position and is proportional to its distance from mean position.

Let a body having simple harmonic motion is represented by the equation

$$x = A \sin\omega t \text{ --------(1)}$$

$$\dot{x} = A\omega \cos\omega t \text{ -------(2)}$$

$$\ddot{x} = -A\omega^2 \sin\omega t \text{ --------(3)}$$

$$\text{or } \ddot{x} = -\omega^2 x \text{ --------(4)}$$

where x, $\dot{x}$ and $\ddot{x}$ represent the displacement, velocity and acceleration of the body respectively.

Phase Difference: Suppose there are two vectors $x_1$and $x_2$having frequencies ωrad/sec each. The vibrating motions can be expressed as

$$x_1 = A_1 \sin\omega t$$

$$x_2 = A_2 \sin(\omega t + \emptyset) \text{ ------(5)}$$

Resonance: When the frequency of external excitation is equal to the natural frequency of a vibrating body, the amplitude of vibration becomes excessively large. This concept is known as resonance.

Mechanical Systems: The systems consisting of mass, stiffness and damping are known as mechanical systems.

Methods of Vibration Analysis

Some of the methods of vibration analysis are discussed here;

Energy Method:

According to this method the sum of the energies associated with the system is constant;

i.e., Kinetic Energy + Potential Energy = Constant, or (K.E. + P.E.) = Constant

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

$m\dot{x}\ddot{x} + kx\dot{x} = 0$

$$\text{or } m\ddot{x} + kx = 0 \text{ ---------(6)}$$

This is the equation of motion.

If the motion is simple harmonic given as,

$$x = A \sin \omega t$$

$$\text{So, } \ddot{x} = - A\,\omega^2 \sin \omega t$$

$$\text{Then } - mA\,\omega^2 \sin \omega t + kA \sin \omega t = 0 \text{ ---------(7)}$$

$$\text{Thus, } \omega = \sqrt{\frac{k}{m}} \text{ rad/sec, or } f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz -------(8)}$$

Rayleigh's Method: This method is the extension of energy method. The method is based on the principle that the total energy of a vibrating system is equal to the maximum potential energy.

At any moment total energy is either the kinetic energy or potential energy or the sum of the both. Let us say that the total energy is kinetic energy which is expressed as,

$$(K.E.)_{max} = (\tfrac{1}{2} m\dot{x}^2)_{max} = \tfrac{1}{2} m(\omega A)^2$$

$$(P.E.)_{max} = (\tfrac{1}{2} kx^2)_{max} = \tfrac{1}{2} kA^2$$

So, $m(\omega A)^2 = kA^2$

$$m\omega^2 = k$$

$$\omega = \sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz --------(9)}$$

Equilibrium Method: According to this method the algebraic sum of the forces and moments acting on the system must be zero. If the external force acting on the system is F, spring force kx, damping force c$\dot{x}$ and inertia force m$\ddot{x}$, then the equation of motion can be written as

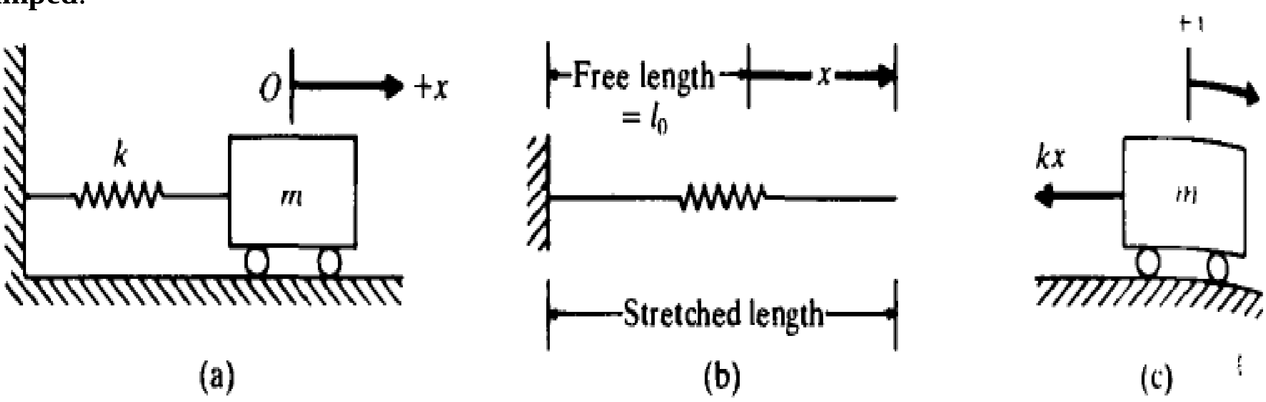
$$m\ddot{x} + c\dot{x} + kx = F \text{ --------(10)}$$

FREE VIBRATION OF SINGLE DEGREE OF FREEDOM SYSTEMS

Introduction

Figure shows a spring-mass system that represents the simplest possible vibratory system. It is called a single degree of freedom system, since one coordinate (x) is sufficient to specify the position of the mass at any time. There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be a free vibration. Since there is no element that causes dissipation of energy during the motion of the mass, the amplitude of motion remains constant with time; it is an **undamped system**.

In actual practice, except in a vacuum, the amplitude of free vibration diminishes gradually over time, due to the resistance offered by the surrounding medium (like air). Such vibrations are said to be **damped**.



Spring-mass System in Horizontal Position: Consider the undamped single degree of freedom system shown in figure (previous slide). The mass supported on frictionless rollers and can have translatory motion in the horizontal direction. The unstretched length of the spring is l_0. Let the mass be displaced a distance +x from its rest position. This results in a spring force kx, as shown in figure. Newton's second law states that, mass X acceleration = Resultant force on the mass ------(1)

The applications of equation 1 to the mass **m** yields the equation of motion,

$$m\ddot{x} = - kx$$

$$\text{or, } m\ddot{x} + kx = 0 \text{ -------(2)}$$

Where $\ddot{x} = \frac{d^2x}{dt^2}$ is the acceleration of the mass.

Spring-mass System in Vertical Position: Consider the configuration of the spring-mass system shown in figure (in next slide). The mass hangs at the lower end of a spring, which in turn is attached to a rigid support at its upper end. At rest the mass will hang in a position called the static equilibrium position, in which the upward spring force exactly balances the downward gravitational force on the mass. In this position the length of the spring is $\boldsymbol{l_0 + \delta_{st}}$, where $\boldsymbol{\delta_{st}}$ is the static deflection – the elongation due to the weight **W** of the mass **m**. from figure, we find that, forstatic equilibrium,

$$W = mg = k\delta_{st} \text{ -------(3)}$$

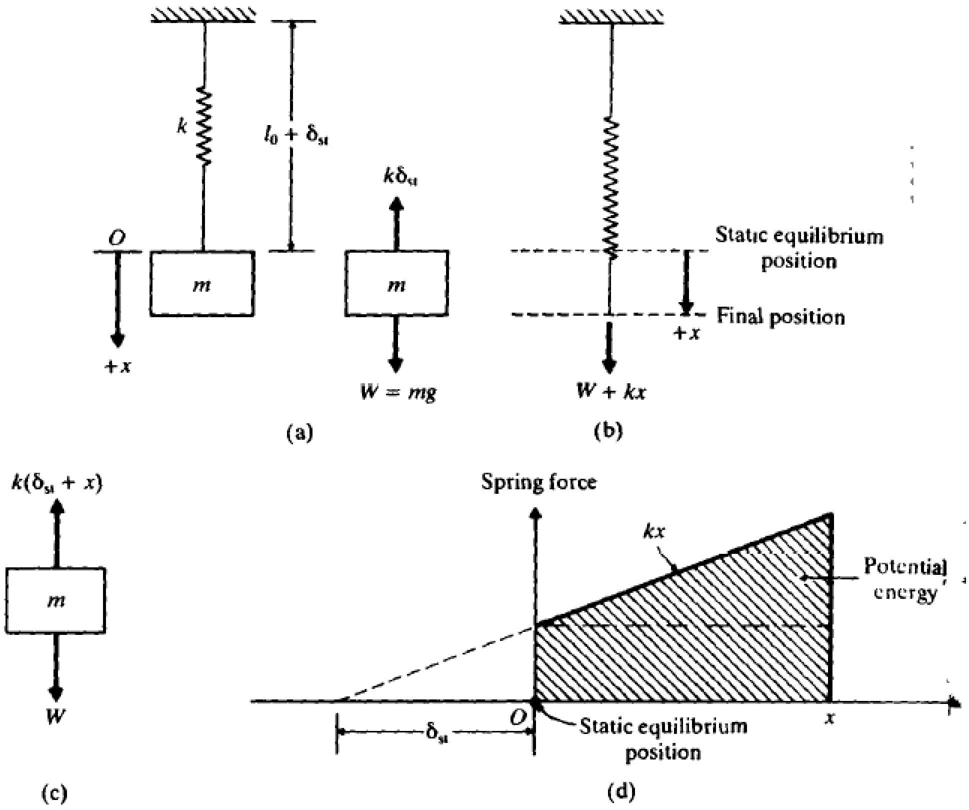
Where g, is the acceleration due to gravity. Let the mass be deflected a distance **+x**; from its static equilibrium position; thenthe spring force is **– k(x +** $\boldsymbol{\delta_{st}}$**)**, as shown in figure. The applicationof Newton's second law of motion to the mass **m** gives

$$m\ddot{x} = – k(x + \delta_{st}) + W$$

and since $k\delta_{st}$= W, we obtain

$$m\ddot{x} + kx = 0 \text{ --------(4)}$$

Equations 2 and 4 are identical. This indicates that when a mass moves in a vertical direction, we can ignore its weight, provided we measure x from its static equilibrium position.



Equation 2 can also be derived by using the conservation of energy principle. To apply this principle, first note that the system shown in figure is conservative, since there is no energy dissipation due to damping. During vibration, the energy of the system is partly kinetic and partly potential. The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy U is stored in the spring by virtue of its elastic deformation. Due to the conservation of energy, we have

T + U = constant

or $\frac{d}{dt}(T + U)$ = 0 --------(5)

The kinetic and potential energies are given by

T = $\frac{1}{2}m\dot{x}^2$ ---------(6)

and U = $\frac{1}{2}$kx^2 --------- (7)

Substitutions of equation 6 and 7 into equation 5 yields the desired equation

m$\ddot{x}$ + kx = 0 ------(2)

The solution of equation 2 can be found by assuming

x(t) = Ce^{st} -------- (8)

where C and s are constants to be determined. Substitution of equation 8 into equation 2 gives

C(ms^2 + k) = 0

Since C cannot be zero, we have

ms^2 + k = 0 ------(9)

And hence, s = $\pm(-\frac{k}{m})^{1/2}$ = $\pm i\omega_n$ -------- (10)

where i = $(-1)^{1/2}$ and,

$\omega_n = (\frac{k}{m})^{1/2}$ ----------(11)

Equation 9 is called the auxiliary or the characteristic equation corresponding to the differential equation 2. the two values of ***s*** given by equation 10 are the roots of the characteristic equation, also known as the eigen values or the characteristic values of the problem. Since both values of ***s*** satisfy equation 9, the general solution of equation 2 can be expressed as

x(t) = $C_1e^{i\omega_n t} + C_2e^{-i\omega_n t}$ --------(12)

where $C_1$and C_2 are constants. By using the identities

$e^{\pm i\alpha t}$ = $\cos\alpha t \pm i\sin\alpha t$

Equation 12 can be written as x(t) = $A_1\cos\omega_n t$ + $A_2\sin\omega_n t$ --------(13)

where A_1 and A_2 are new constants. The constants C_1 and C_2 or $A_1$and A_2 can be determined from the initial conditions of the system. If the values of displacement x(t) and velocity $\dot{x}$(t) = $\frac{dx}{dt}(t)$ are specified as x_0 and $\dot{x_0}$ at t = 0, we have, from equation 13,

x(t = 0) = A_1 = x_0

$\dot{x}$(t = 0) = - $\omega_n A_2$ = $\dot{x}_0$ ---------(14)

Hence, A_1 = x_0 and A_2 = $\frac{\dot{x}_0}{\omega_n}$. Thus, the solution of equation 2 subject to the initial conditions of equations 14 is given by

x(t) = $x_0\cos\omega_n t$ + $\frac{\dot{x}_0}{\omega_n}$ $\sin\omega_n t$ --------(15)

Equations 12, 13 and 15 are harmonic functions of time. The motion is symmetric about the equilibrium position of the mass **m**. The velocity is a maximum and the acceleration is zero each time the mass passes through this position. At the extreme displacements the velocity is zero and the acceleration is a maximum. Since this represents simple harmonic motion, the spring-mass system itself is called a harmonic oscillator. The quantity ω_n, given by 11, represents the natural frequency of vibration of the system.

Equation 13 can be expressed in a different form by introducing the notation

$$A_1 = A\cos\emptyset$$
$$A_2 = A\sin\emptyset \text{ --------(16)}$$

Where A and $\emptyset$ are the new constants which can be expressed in terms of A_1 and A_2 as

$$A = (A_1^2 + A_2^2)^{1/2} = [x_0^2 + (\frac{\dot{x}_0}{\omega_n})^2]^{1/2} = \text{Amplitude}$$
$$\emptyset = \tan^{-1}(\frac{A_2}{A_1}) = \tan^{-1}(\frac{\dot{x}_0}{x_0\omega_n}) = \text{Phase angle --------(17)}$$

Introducing equation 16 into 13, the solution can be written as

$$x(t) = A\cos(\omega_n t - \emptyset) \text{ -------- (18)}$$

Note the following aspects of the spring-mass system:

1] If the spring-mass system is in a vertical position, the circular natural frequency can be expressed as

$$\omega_n = (\frac{k}{m})^{1/2} \text{ --------(19)}$$

The spring constant k can be expressed in terms of the mass ***m*** from equation 19 as

$$k = \frac{W}{\delta_{st}} = \frac{mg}{\delta_{st}} \text{ --------(20)}$$

Substitution of equation 20 into 11 yields,

$$\omega_n = (\frac{g}{\delta_{st}})^{1/2} \text{ --------(21)}$$

Hence the natural frequency in cycles per second and the natural period are given by,

$$f_n = \frac{1}{2\pi}(\frac{g}{\delta_{st}})^{1/2} \text{ ---------(22)}$$
$$\tau_n = \frac{1}{f_n} = 2\pi(\frac{g_{st}}{g})^{1/2} \text{ --------(23)}$$

Thus, when the mass vibrate in a vertical direction. We can compute the natural frequency and the period of vibration by simply measuring the static deflection δ_{st}. It is not necessary that we know the spring stiffness k and the mass m.

2] from equation 18, the velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ of the mass ***m*** at time ***t*** can be obtained as;

$$\dot{x}(t) = \frac{dx}{dt}(t) = -\omega_n A\sin(\omega_n t - \emptyset) = \omega_n A\cos(\omega_n t - \emptyset + \frac{\pi}{2})$$
$$\ddot{x}(t) = \frac{d^2x}{dt^2}(t) = -\omega_n^2 A\cos(\omega_n t - \emptyset) = \omega_n^2 A\cos(\omega_n t - \emptyset + \pi) \text{ --------(24)}$$

Equation 24 shows that the velocity leads the displacement by $\frac{\pi}{2}$ and the acceleration leads the displacement by π.

3] if the initial displacement (x_0) is zero, equation 24 becomes

$$x(t) = \frac{\dot{x}_0}{\omega_n}\cos\left(\omega_n t - \frac{\pi}{2}\right) = \frac{\dot{x}_0}{\omega_n} sin(\omega_n t) \text{ ---------(25)}$$

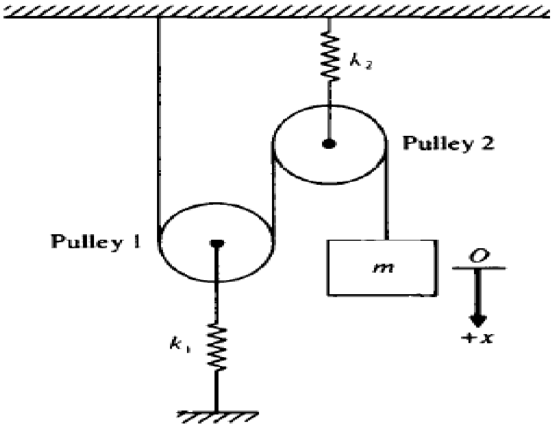
On the other hand, if the initial velocity $(\dot{x}_0)$ is zero, the solution becomes;

$$x(t) = x_0 cos\omega_n t \text{ --------(26)}$$

Problem] Determine the natural frequency of the system shown in figure. Assume the pulleys to be frictionless and of negligible mass.

Solution: since the pulleys are frictionless and massless, the tension in the rope is constant and is equal to the weight W of the mass m. thus, the upward force acting on pulley 1 is 2W, and the downward force acting on pulley 2 is 2W. The center of pulley 1 moves up by a distance 2W/ k_1 and the center of pulley 2 moves down by 2W/k_2. Thus, the total movements of the mass m is;

$$2(\frac{2W}{k_1} + \frac{2W}{k_2})$$



as the rope on either side of the pulley is free to move the mass downward. If k_{eq} denotes the equivalent spring constant of the system;

$$\frac{weight\ of\ the\ mass}{equivalent\ spring\ constant} = \text{net displacement of the mass}$$

$$\frac{W}{k_{eq}} = 4W\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{4W(k_1 + k_2)}{k_1 k_2}$$

$$k_{eq} = \frac{k_1 k_2}{4(k_1 + k_2)}$$

If the equation of motion of the mass is written as

$$m\ddot{x} + k_{eq}x = 0$$

The natural frequency is given by

$$\omega_n = \left(\frac{k_{eq}}{m}\right)^{1/2} = \left(\frac{k_1 k_2}{4m(k_1 + k_2)}\right)^{1/2} \text{ rad/sec}$$

$$\text{or, } f_n = \frac{\omega_n}{2\pi} = \frac{1}{4\pi}\left[\frac{k_1 k_2}{m(k_1 + k_2)}\right]^{1/2} \text{ cycles/sec}$$

Free Vibration of an Undamped Torsional System

If a rigid body oscillates about a specific reference axis, the resulting motion is called. **torsional vibration.** In this case displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple.

Figure shows a disc, which has a polar mass moment of inertiaJ_0, mounted at one end of solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be θ; θalso represents the angle of twist of the shaft. From the theory of torsion of circular shafts, we have the relation

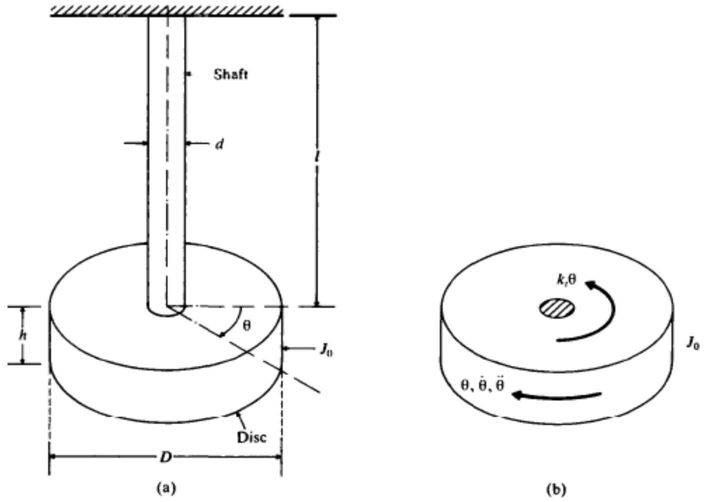
$$M_t = \frac{GJ\theta}{l} \text{ --------(27)}$$

WhereM_t is the torque that produces the twist θ, G is the shear modulus, l is the length of the shaft, J is the polar moment of inertia of the cross section of the shaft given by

$$J = \frac{\pi d^4}{32} \text{ --------(28)}$$

And d is the diameter of the shaft. If the disc is displaced by θ from its equilibrium position, the shaft provides a restoring torque of magnitudeM_t. Thus the shaft acts as a torsional spring with a torsional spring constant.

$$k_t = \frac{M_t}{\theta} = \frac{GJ}{l} = \frac{\pi G d^4}{32 l} \text{ ------- (29)}$$



The equation of angular motion of the disc about its axis can be derived by using Newton's second law or the principle of conservation of energy. By considering the free body diagram of the disc, we can derive the equation of motion by applying Newton's second law of motion:

$$J_0\ddot{\theta} + k_t\theta = 0 \text{ --------(30)}$$

Which can be seen to be identical to equation 2 if the polar mass moment of inertia J_0 the angular displacement θ, and the torsional spring constant k_t are replaced by the mass m, the displacement x, and the linear spring constant k, respectively. Thus the natural circular frequency of the torsional system is;

$$\omega_n = \left(\frac{k_t}{J_0}\right)^{1/2} \text{ -------(31)}$$

And the period and natural frequency of vibration in cycles per second are

$$\tau_n = 2\pi\left(\frac{J_0}{k_t}\right)^{1/2} \text{ --------(32)}$$

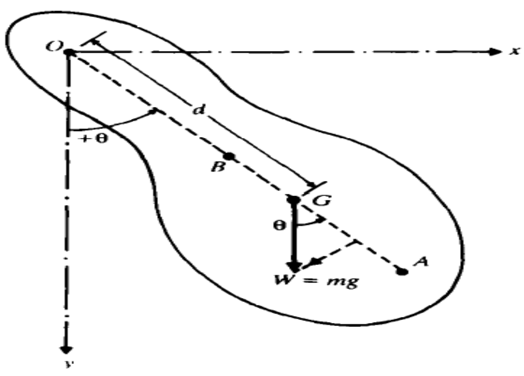
$$f_n = \frac{1}{2\pi}\left(\frac{k_t}{J_0}\right)^{1/2} \text{ ---------(33)}$$

Problem] Any rigid body pivoted at a point other than its center of mass will oscillate about the pin point under its own gravitational force. Such a system is known as a compound pendulum (as shown in figure). Find the natural frequency of such a system.

Solution: Let O be the point of superposition and G be the center of mass of the compound pendulum, as shown in figure. Let the rigid body oscillate in the xy plane so that coordinate θ can be used to describe its motion. Let d denote the distance between O and J_0 the mass moment of inertia of the body about the z-axis (perpendicular to both x and y). For a displacement θ, the restoring torque (due to the weight of the body W) (W d sinθ) and the equation of motion is

$$J_0\ddot{\theta} + Wd\,sin\theta = 0 \text{ --------(34)}$$

For a small angles of oscillation, sin $\theta = \theta$, hence equation can be expressed as



$$J_0\ddot{\theta} + Wd\,\theta = 0 \text{ --------(35)}$$

This gives the natural frequency of the compound pendulum:

$$\omega_n = \left(\frac{Wd}{J_0}\right)^{1/2} = \left(\frac{mgd}{J_0}\right)^{1/2} \text{ --------(36)}$$

Comparing equation 36 with the natural frequency of a simple pendulum, $\omega_n = (g/l)^{1/2}$, we can find the length of the equivalent simple pendulum:

$$l = \frac{J_0}{md} \text{ --------- (37)}$$

If J_0 is replaced by mk_0^2 where k_0 is the radius of gyration of the body about O, equation 36 and 37 becomes

$$\omega_n = \left(\frac{gd}{k_0^2}\right)^{1/2} \text{ --------(38)}$$

$$l = \left(\frac{k_0^2}{d}\right) \text{ --------- (39)}$$

If k_G denotes the radius of gyration of the body about G, we have

$$k_0^2 = k_G^2 + d^2 \text{ ------- (40)}$$

and equation 39 becomes,

$$l = \left(\frac{k_G^2}{d} + d\right) \text{--------(41)}$$

If the line OG is extended to point A such that

$$GA = \frac{k_G^2}{d} \text{ ------- (42)}$$

Equation 41 becomes, $l = GA + d = OA$ -------- (43)

Hence, from equation 35, ω_n is given by,

$$\omega_n = \left\{\frac{g}{k_0^2/d}\right\}^{1/2} = \left(\frac{g}{l}\right)^{1/2} = \left(\frac{g}{OA}\right)^{1/2} \text{ ------- (44)}$$

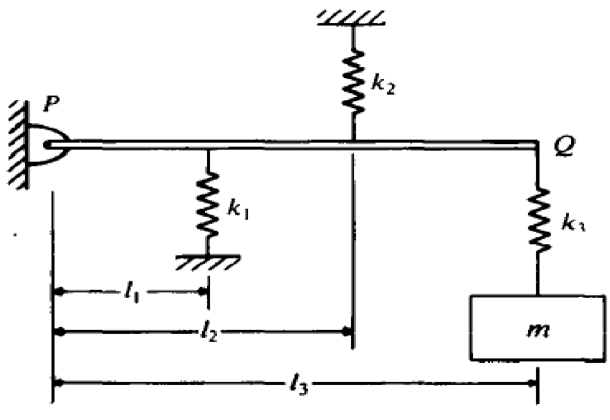
This equations shows that, no matter whether the body is pivoted from O and A, its natural frequency is the same. The point A is called the ***center of percussion***.

Problem Based on Unit 1

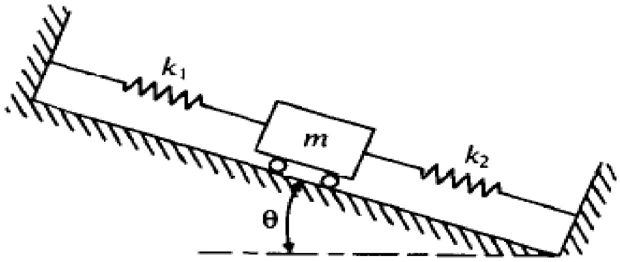
1] A spring-mass system has a natural period of 0.21 sec. what will be the new period if the spring constant is (i) increased by 50% and (ii) decreased by 50%?

2] A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the frequency is altered by 45%. Find the mass and spring constant of the original system.

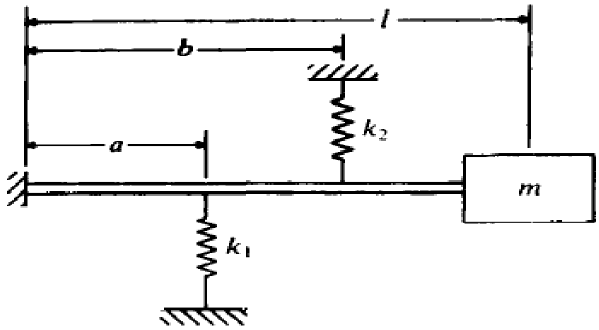
3] Three springs and a mass are attached to a rigid, weightless, bar PQ as shown in figure. Find the natural frequency of vibration of the system.



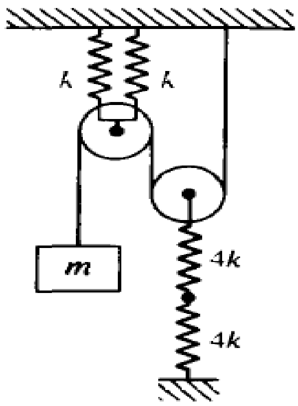
4] Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane, as shown in figure.



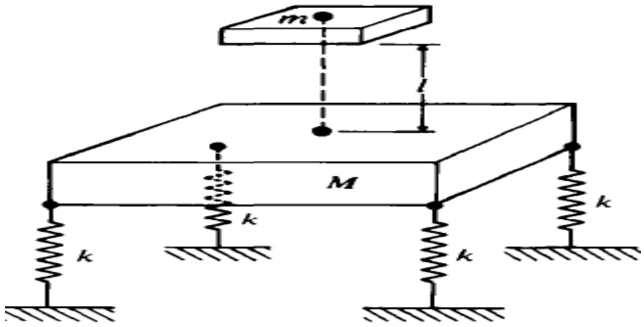
5] Find the natural frequency of the system shown in figure with and without the springs k_1 and k_2 in the middle of the elastic beam.



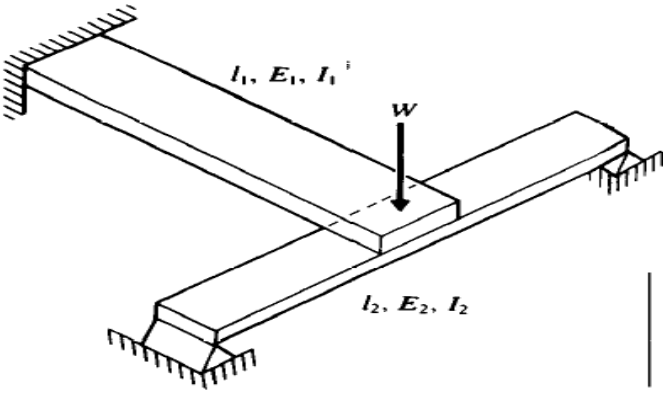
6] Find the natural frequency of the pulley system shown in figure by neglecting the friction and the masses of the pulleys.



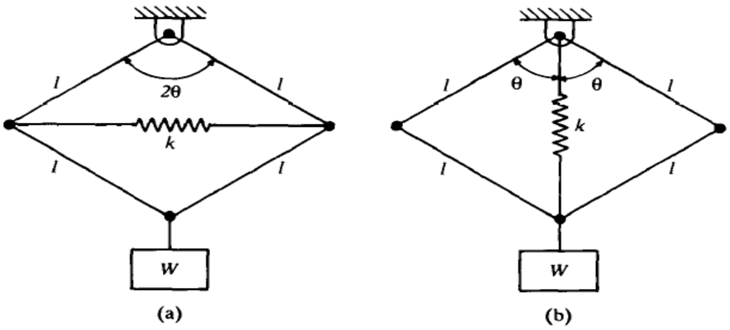
7] A rigid block of mass M is mounted on four elastic supports as shown in figure. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each elastic support is k, find the natural frequency of vibration of the system (a) without the mass m, and (b) with the mass m. Also, find the resulting motion of the system in case (b).



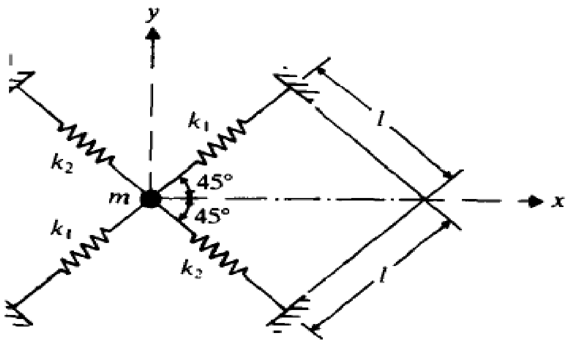
8] Derive the expression for the natural frequency of the system shown in figure. Note that the load W is applied at the tip of beam 1 and midpoint of beam 2.



9] The natural frequency of a spring-mass system is found to be 2 Hz. When an additional mass of 1 kg is added to the original mass m, the natural frequency is reduced to 1 Hz. Find the spring constant k and the mass m.



10] Four weightless rigid links and a spring are arranged to support a weight W in two different ways as shown in figure. Determine the natural frequencies of vibration of the two arrangement.



11] Figure shows a small mass 'm' restrained by four linearly elastic springs, each of which has an unstretched length *l,* and an angle of orientation of 45° with respect to the x-axis. Determine the equation of motion for small displacements of the mass in the x-direction.

UNIT-2

HARMONICALLY EXITED VIBRATIONS

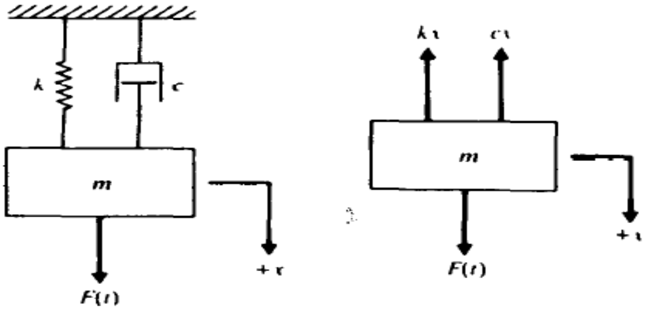
Introduction

A dynamic system is often subjected to some type of external force or excitation, called the *forcing* or *exciting function*. This excitation is usually time-dependent. It may be *harmonic*, *non-harmonic* but *periodic*, *non-periodic*, or *random* in nature. The response of a system to a harmonic excitation is called *harmonic response*. The non-periodic excitation may have a long or short duration. The response of a dynamic system to suddenly applied non-periodic excitations is called *transient response*.

Let us suppose the dynamic response of a single degree of freedom system under harmonic excitations of the form F(t) = $F_0 e^{i(\omega t+\emptyset)}$ or F(t) = $F_0 \cos(\omega t+\emptyset)$ or F(t) = $F_0 \,\mathrm{Sin}(\omega t+\emptyset)$, where F_0 is the amplitude, ω is the frequency, and $\emptyset$ is the phase angle of the harmonic excitation. The value of $\emptyset$ depends on the value of F(t) at t = 0 and is usually taken to be zero. Under a harmonic excitation, the response of the system will also be harmonic. If the frequency of excitation coincides with the natural frequency of the system, the response of the system will be very large. This condition, known as *resonance*, is to be avoided to prevent failure of the system.

Equation of Motion

If a force F(t) acts on a viscosity damped spring-mass system as shown in figure. The equation of motion can be obtained using Newton's second Law:



(a) (b) Free body diagram

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ -------(1)}$$

Since this equation is homogeneous, its general solution x(t) is given by the sum of the homogeneous solution, $x_h(t)$, and the particular solution, $x_p(t)$.

The homogeneous solution, which is the solution of the homogeneous equation;

$$m\ddot{x} + c\dot{x} + kx = 0 \text{ ---------(2)}$$

represents the free vibration of the system. This free vibration dies out with time under each of the three possible conditions of damping (underdamping, critical damping and overdamping), and under all possible initial conditions. Thus, the general solution of equation 1 eventually reduces to the particular solution $x_p(t)$, which represents the steady-state vibration. The steady-state motion is present as long as the forcing function is present. The vibrations of homogeneous, particular, and general solutions with time for a typical case are shown in figure 2. it can be seen that $x_h(t)$ dies out and x(t) becomes $x_p(t)$ after some time (τ in figure 2). The part of the motion that dies out due to damping (the free vibration part) is called transient. The rate at which the transient motion decays depends on the values of the system parameters k, c, and m.

Response of an Undamped System under Harmonic Force

First we consider an undamped system subjected to a harmonic force, for the sake of simplicity. If a force F(t) = $F_0 cos\omega t$ acts on the mass m of an undamped system, the equation of motion is,

$$m\ddot{x} + \text{kx} = F_0 cos\omega t \text{ ------- (3)}$$

The homogeneous solution of this equation is given by,

$$x_h(t) \approx C_1 cos\omega_n t + C_2 sin\omega_n t \text{ ------- (4)}$$

where $\omega_n = (k/m)^{1/2}$ is the natural frequency of the system. Because the exciting force F(t) is harmonic, the particular solution $x_p(t)$ is also harmonic and has the same frequency ω. Thus, we assume a solution in the form

$$x_p(t) = \text{X}cos\omega t \text{ -------(5)}$$

where X is a constant that denotes the maximum amplitude of $x_p(t)$. By substituting equation 5 into 3 and solving for X, we obtain,

Case 1. when 0 <ω/ω_n< 1, the denominator in equation 10 is positive and the response is given by equation 5 without change. The harmonic response of the system $x_p(t)$is said to be in phase with the external force as shown in figure.

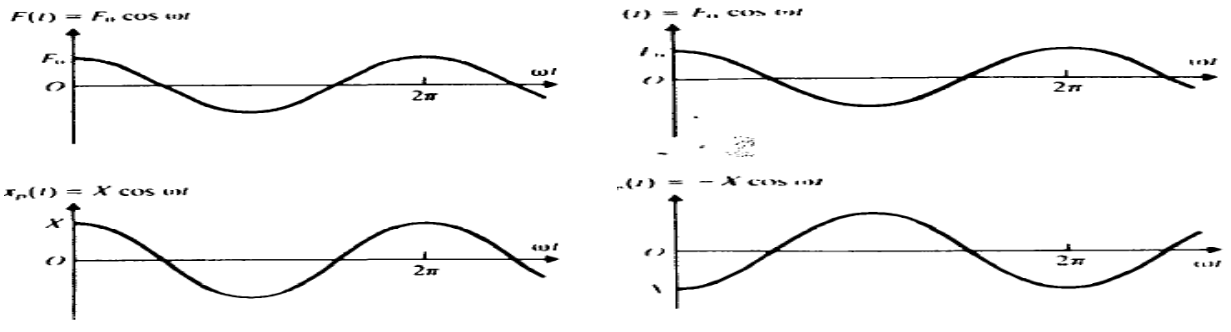
Case 2. when ω/ω_n> 1, the denominator in equation 10 is negative, and the steady-state solution can be expressed as $x_p(t)$ = - Xcosωt -------- (11)

where the amplitude of motion X is redefined to be a positive quantity as

$$\text{X} = \frac{\delta_{st}}{(\frac{\omega}{\omega_n})^2 - 1} \text{ ------ (12)}$$

The variations of F(t) and $x_p(t)$with time are shown in figure. Since $x_p(t)$and F(t) have opposite signs, the response is said to be 180° out of phase with the external force. Further, as $\omega/\omega_n \rightarrow \alpha$, X → 0. thus, the response of the system to a harmonic force of very high frequency is close to zero.

Case 3. when $\omega/\omega_n \rightarrow 1$, the amplitude X given by equation 10 or 12 becomes infinite. The condition, for which the forcing frequency $\omega$is equal to the natural frequency of the system ω_n, is called resonance. To find the response for this



condition. We rewrite equation as

$$\text{x(t)} = x_0 cos\omega_n t + \frac{\dot{x}_0}{\omega_n} sin\omega_n t + \delta_{st}[\frac{cos\omega t - cos\omega_n t}{1 - (\frac{\omega}{\omega_n})^2}] \text{ -------- (13)}$$

Since the last term of the equation takes an indefinite form for $\omega = \omega_n$, we apply L-Hospital's rule to evaluate the limit of this term;

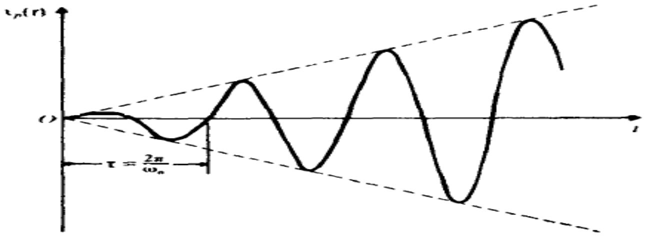
$$\lim_{\omega \to \omega_n} [\frac{cos\omega t - cos\omega_n t}{1 - (\frac{\omega}{\omega_n})^2}] = \lim_{\omega \to \omega_n} [\frac{\frac{d}{d\omega}(cos\omega t - cos\omega_n t)}{\frac{d}{d\omega}\{1 - \left(\frac{\omega}{\omega_n}\right)^2\}}] = \lim_{\omega \to \omega_n} [\frac{tsin\omega t}{2\frac{\omega}{\omega_n^2}}] = \frac{\omega_n t}{2} sin\omega_n t \text{ -------- (14)}$$

Thus the response of the system at resonance becomes;

$$x(t) = x_0 cos\omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st}\omega_n t}{2} \sin \omega_n t \text{ -------- (15)}$$

It can be seen from equation 15 that at resonance, x(t) increases indefinitely.

The last term of equation 15 is shown in figure from which the amplitude of the response can be seen to increase linearly with time.



Response of a damped system under harmonic force

If the forcing function is given by F(t) = $F_0 cos\omega t$, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 cos\omega t \text{ ------- (16)}$$

The particular solution of equation 16 is also expected to be harmonic; we assume it in the form

$$x_p(t) = Xcos(\omega t - \emptyset) \text{ ------- (17)}$$

Where X and $\emptyset$ are constants to be determined. X and $\emptyset$ denote the amplitude and phase angle of the response, respectively.

$$X[(k - m\omega^2)\cos(\omega t - \emptyset) - c\omega\sin(\omega t - \emptyset)] = F_0 cos\omega t \text{ -------- (18)}$$

Using the trigonometric relations;

$$\cos(\omega t - \emptyset) = cos\omega t cos\emptyset + sin\omega t sin\emptyset$$
$$\sin(\omega t - \emptyset) = sin\omega t cos\emptyset - cos\omega t sin\emptyset$$

In equation 18 and equating the coefficients of $cos\omega t$ and $sin\omega t$ on both sides of the resulting equation, we obtain

$$X[(k - m\omega^2)\cos\emptyset + c\omega\sin\emptyset] = F_0$$
$$X[(k - m\omega^2)\sin\emptyset - c\omega\cos\emptyset] = 0 \text{ -------- (19)}$$

Solution of equation 19 gives, $\quad X = \frac{F_0}{[(k - m\omega^2) + c^2\omega^2]^{1/2}}$ -------- (20)

and $\quad \emptyset = \tan^{-1}(\frac{c}{k - m\omega^2})$ --------- (21)

By inserting the expressions of X and $\emptyset$ from equations 20 and 21 into equation 17. We obtain the particular solution of equation 16. figure shows typical plots of the forcing function and (steady-state) response. Dividing both the numerator and denominator of equation 20 by k and making the following substitutions,

$\omega_n = \sqrt{\frac{k}{m}}$ = undamped natural frequency.

$$\zeta = \frac{c}{c_1} = \frac{c}{2m\omega_n};\ \frac{c}{m} = 2\zeta\omega_n,$$

$$\delta_{st} = \frac{F_0}{k} = deflection\ under\ the\ static\ force\ F_0;\ and$$

$$r = \frac{\omega}{\omega_n} = frequency\ ratio$$

We obtain,

$$\frac{X}{\delta_{st}} = \frac{1}{\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + [2\zeta\frac{\omega}{\omega_n}]^2\}^2} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \text{ -------- (22)}$$

and $\quad \emptyset = \tan^{-1}\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - (\frac{\omega}{\omega_n})^2}\} = \tan^{-1}(\frac{2\zeta r}{1 - r^2})$ --------- (23)

As stated, the quantity X/δ_{st} is called the magnification factor, amplification factor, or amplitude ratio.

The variations of X/δ_{st} and $\emptyset$ with the frequency ratio r and the damping ratio ζ are shown in figure. The following observations can be made from (22) and (23).

1] For an undamped system (ζ = 0), equation 23 shows that the phase angle $\emptyset$ = 0 (for r < 1) or 180˚ (for r > 1) and equation 22 reduces.

2] The damping reduces the amplitude ratio for all values of the forcing frequency.

3] The reduction of the amplitude ratio in the presence of damping is very significant at or near resonance.

4] with damping, the maximum amplitude ratio occurs when $r = \sqrt{1-2\zeta^2}$ or $\omega = \omega_n\sqrt{1-2\zeta^2}$ -------- (24)

Which is lower than the undamped natural frequency ω_n and the damped natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

5] The maximum value of X (when r = $\sqrt{1-2\zeta^2}$) is given by

$$(\frac{X}{\delta_{st}})_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \text{ -------- (25)}$$

And the value of X at $\omega = \omega_n$ by $(\frac{X}{\delta_{st}})_{\omega=\omega_n} = \frac{1}{2\zeta}$ -------- (26)

Equation 25 can be used for the experimental determination of the measure of damping present in the system. In a vibration test, if the maximum amplitude of the response $(X)_{max}$ is measured, the damping ratio of the system can be found usingequation 26. conversely, if the amount of damping is known, one can make an estimate of the maximum amplitude of vibration.

6] For ζ> 1/$\sqrt{2}$. The graph of X has no peaks and for ζ = 0, there is a discontinuity at r = 1.

7] The phase angle depends on the system parameters m, c, and k and the forcing frequency ω but not on the amplitude F_0 of the forcing function.

8] The phase angle $\emptyset$ by which the response x(t) or X lags the forcing function F(t) or F_0 will be very small for small values of r. For very large values of r, phase angle approaches 180˚ asymptotically. Thus, the amplitude of vibration will be in phase with the exciting force for r << 1 and out of phase for r >> 1. The phase angle at resonance will be 90˚ for all values of damping (ζ).

9] Below resonance ($\omega < \omega_n$), the phase angle increases with increase in damping. Above resonance ($\omega > \omega_n$), the phase angle decreases with increase in damping.

Forced Vibration with Coulomb Damping

For a single degree of freedom system with Coulomb or dry friction damping, subjected to a harmonic force F(t) = $F_0 sin\omega t$, the equation of motion is given by

$$m\ddot{x} + kx \pm \mu N = F(t) = F_0 sin\omega t \text{ -------- (27)}$$

Where the sign of the friction force (μN) is positive (negative) when the mass moves from left to right (right to left). The exact solution of equation 27 is quite involved. However, we can expect that if the dry friction damping force is large, the motion of the mass will be discontinuous. On the other hand, if the dry friction force is small compared to the amplitude of the applied force F_0, the steady state solution is expected to be nearly harmonic. In this case, we can find an approximate solution of equation 27 by finding an equivalent viscous damping ratio. To find an equivalent viscous damping ratio, we equate the energy dissipated due to dry friction to the energy dissipated by an equivalent viscous damper during a full cycle of motion. If the amplitude of motion is denoted as X, the energy dissipated by the friction force μN in a quarter cycle is μNX. hence, in a full cycle, the energy dissipated by dry friction damping is given by

$$\Delta W = 4\mu NX \text{ ---------(28)}$$

If the equivalent viscous damping constant is denoted as c_{eq}, the energy dissipated during a full cycle will be

$$\Delta W = \pi c_{eq} \omega X^2 \text{ ---------(29)}$$

By equating equations 28 and 29, we obtain

$$c_{eq} = \frac{4\mu N}{\pi \omega X} \text{ -------(30)}$$

Thus, the steady-state response is given by

$$x_p(t) = X sin(\omega t - \emptyset) \text{ -------(31)}$$

Where the amplitude X can be found from equation:

$$X = \frac{F_0}{[(k - m\omega^2 + (c_{eq}\omega)^2)]^{1/2}} = \frac{(F_0/k)}{[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + (2\zeta_{eq}\frac{\omega}{\omega_n})^2]^{1/2}} \text{ ------- (32)}$$

With

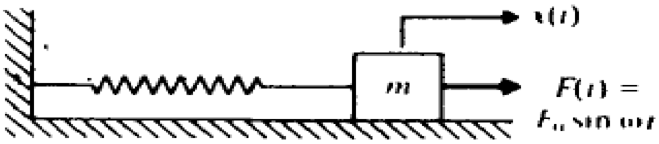
$$\zeta_{eq} = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{2m\omega_n} = \frac{4\mu N}{2m\omega_n \pi \omega X} = \frac{2\mu N}{\pi m \omega \omega_n X} \text{ --------(33)}$$

Substitution of equation 33 into equation 32 gives,

$$X = \frac{(F_0/k)}{[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + (\frac{4\mu N}{\pi k X})^2]^{1/2}} \text{ --------(34)}$$

The solution of this equation gives the amplitude X as,

$$X = \frac{F_0}{k}\left[\frac{1 - (\frac{4\mu N}{\pi F_0})^2}{(1 - \frac{\omega^2}{\omega_n^2})^2}\right]^{1/2} \text{ --------- (35)}$$



As stated earlier, equation 35 can be used only if the friction force is small compared to F_0. In fact, the limiting value of the friction force μN can be found from equation 35. to avoid imaginary values of X, we need to have

$$1 - \left(\frac{4\mu N}{\pi F_0}\right)^2 > 0 \; or \; \frac{F_0}{\mu N} > \frac{4}{\pi}.$$

If this condition is not satisfied, the exact analysis, is to be used. The phase angle $\emptyset$ appearing in equation 31 can be found using, $\emptyset = \tan^{-1}\left(\frac{c_{eq}\omega}{k - m\omega^2}\right) = \tan^{-1}\left[\frac{2\zeta_{eq}\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}\right] = \tan^{-1}\{\frac{\frac{4\mu N}{\pi k X}}{1 - \frac{\omega^2}{\omega_n^2}}\}$ ------- (36)

substituting equation 35 into equation 36 for X, we obtain

$$\emptyset = \tan^{-1}\{\frac{\frac{4\mu N}{\pi F_0}}{[1 - (\frac{4\mu N}{\pi F_0})^2]^{1/2}}\} \text{ -------(37)}$$

Equation 36 shows that $tan\emptyset$ is a constant for a given value of $F_0/\mu N$. $\emptyset$ is discontinuous at $\frac{\omega}{\omega_n} > 1$. thus equation 37 can also be expressed as

$$\emptyset = \tan^{-1}\{\frac{\pm\frac{4\mu N}{\pi F_0}}{[1 - (\frac{4\mu N}{\pi F_0})^2]^{1/2}}\} \text{ -------- (38)}$$

Equation shows that the friction serves to limit the amplitude of forced vibration for $\frac{\omega}{\omega_n} \neq 1$. however, at resonance ($\frac{\omega}{\omega_n} = 1$), the amplitude becomes infinite. This can be explained as follows. The energy directed into the system over one cycle which it is excited harmonically at resonance is

$$\Delta W' = \int_0^{\tau} F\frac{dx}{dt}dt = \int_0^{\tau = 2\pi/\omega} F_0 sin\omega t.[\omega X cos(\omega t - \emptyset)]dt \text{ ------(39)}$$

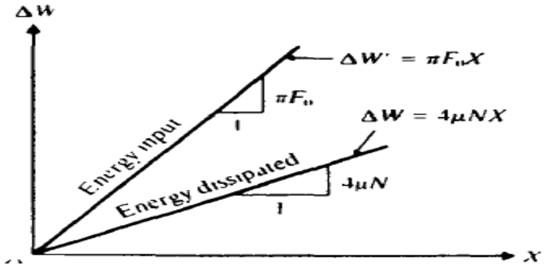
Since equation 36 gives $\emptyset$ = 90° at resonance, equation 39 becomes

$$\Delta W' = F_0 X \omega \int_0^{2\pi/\omega} sin^2 \omega t dt = \pi F_0 X \text{ ---------(40)}$$

The energy dissipated from the system is given by equation. Since $\pi F_0 X > 4\mu NX$ for X to be real-valued. $\Delta W^{'}$>ΔW at resonance (see figure). Thus, more energy isdirected into the system per cycle than is dissipated per cycle. This extra energy is used to build up the amplitude of vibration. For the non resonant condition ($\frac{\omega}{\omega_n} \neq 1$) the energy input can be found from equation (39).

$$\Delta W' = \omega F_0 X \int_0^{2\pi/\omega} sin\omega t cos(\omega t - \emptyset)dt = \pi F_0 X sin\emptyset \text{ -------- (41)}$$

Due to the presence of sin$\emptyset$ in equation 41. the input energy curve in figure is made to coincide with the dissipated energy curve. So the amplitude is limited. Thus, the phase of the motion $\emptyset$ can be seen to limit the amplitude of the motion.
The periodic response of a spring-mass system with Coulomb damping subjected to base excitation.

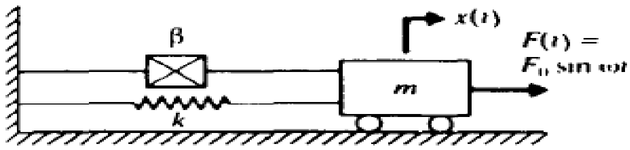


Forced Vibration with Hysteresis Damping

Consider a single degree of freedom system with hysteresis damping and to a harmonic force F(t) = $F_0 sin\omega t$ as indicated in figure. The equation of motion of the mass can be derived by using

$$m\ddot{x} + \frac{\beta k}{\omega}\dot{x} + kx = F_0 sin\omega t \text{ ------(1)}$$

where $\frac{\beta k}{\omega}\dot{x}$ = ($\frac{h}{\omega}$)$\dot{x}$ denotes the damping force. Although the solution of equation is quite involved for a general forcing function F(t), our interest is to find the response under a harmonic force.



Now, The steady-state solution of equation be,

$$x_p(t) = X\ \mathrm{Sin}(\omega t - \emptyset) \text{ ------(2)}$$

But substituting equation (2) into equation (1), we get

$$X = \frac{F_0}{k\{\left(1-\frac{\omega^2}{\omega_n^2}\right)+\beta^2\}^{1/2}} \text{ ------(3)}$$

and, $\emptyset = \tan^{-1}[\frac{\beta}{(1-\frac{\omega^2}{\omega_n^2})}]$ ------(4)

with these equation, we have following points:
1] The amplitude ratio, X/(F_0/k) attains its maximum value of F_0/kβ at the resonant frequency ($\omega = \omega_n$).
2] The phase angle $\emptyset$ has a value of $\tan^{-1}\beta$ at $\omega = 0,$ in the case of hysteresis damping.

3] Also, equation of motion (suppose harmonic excitation is F = $F_0 e^{i\omega t}$)

$$m\ddot{x} + \frac{\beta k}{\omega}\dot{x} + kx = F_0 e^{i\omega t} \text{ -------(5)}$$

In this case, the response x(t) is also a harmonic function involving the factor $e^{i\omega t}$. Hence, $\dot{x}(t)$ is given by i$\omega x(t)$ then equation becomes;

$$m\ddot{x} + \frac{\beta k}{\omega} i\omega x + kx = F_0 e^{i\omega t}$$

$$m\ddot{x} + k(1 + i\beta)x = F_0 e^{i\omega t} \text{ ------(6)}$$

where, the quantity k(1 + iβ) is called the complex stiffness or complex damping. The steady-state solution is given by the real part of,

$$x(t) = \frac{F_0 e^{i\omega t}}{k[\left(1 - \frac{\omega^2}{\omega_n^2}\right) + i\beta]} \text{ -------(7)}$$

*** Hysteresis damping model with a constant stiffness and loss factor is only applicable for harmonic excitation.**

Problem Based on unit 2

Problem 1]: A spring-mass system consists of a mass weighing 100N and a spring with a stiffness of 2000N/m. The mass is subjected to resonance by a harmonic force F(t) = 25cosωt N. Find the amplitude of the forced motion at the end of

(i) 1/4 cycle, (ii) 2(1/2)cycle, (iii) 5(3/4) cycles.

Problem 2]: A spring-mass system with m = 10 kg and k = 5000 N/m is subjected to a harmonic force of amplitude 250 N and frequency ω. If the maximum amplitude of the mass is observed to be 100 mm, find the value of ω.

Problem 3]: A mass 'm' is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force having an amplitude of 100 N and a frequency of 5 Hz. The amplitude of the forced motion of the mass is observed to be 20 mm. find the value of 'm'.

Problem 4]:A weight of 50N is suspended from a spring of stiffness of 4000 N/m and is subjected to a harmonic force of amplitude 60 N and frequency 6 Hz. Find

(i) the extension of the spring due to the suspended weight.

(ii) the static displacement of the spring due to the maximum applied force and

(iii) the amplitude of forced motion of the weight.

Solution:

(i) $\delta = \frac{\omega}{k} = \frac{50}{4000} = 0.0125\, m$

(ii) $\delta_{st} = \frac{F_0}{k} = \frac{60}{4000} = 0.015\, m$

(iii) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000\ X\ 9.81}{50}} = 28.0143\, rad/sec$

$$\omega = 2\pi f = 2X\pi X6 = 37.6992\, rad/sec$$

$$X = \delta_{st}\left[\frac{1}{1 - (\frac{\omega}{\omega_n})^2}\right] = 0.0185\, m$$

Problem 5]: A spring-mass system is subjected to a harmonic force whose frequency is close to the natural frequency of the system. If the forcing frequency is 39.8Hz, determine the period of beating?

Answer: $T\ (period) = \frac{2\pi}{2\pi(f_n - f)} = \frac{10}{2} = 5\ sec$

UNIT-3

VIBRATION UNDER GENERAL FORCING CONDITIONS

Introduction

The vibration of a viscously damped single degree of freedom system under general forcing conditions. If the excitation is periodic but not harmonic, it can be replaced by a sum of harmonic function using the harmonic analysis procedure.

* If the system is subjected to a suddenly applied non-periodic force, the response will be transient, since steady-state vibrations are not usually produced.

Response Under a General Periodic Force

When the external force F(t) is periodic with period $\tau = 2\pi/\omega$, it can be expanded in a Fourier series,

F(t) = $\frac{a_0}{2} + \sum_{j=1}^{\propto} a_j cosj\omega t + \sum_{j=1}^{\propto} b_j sinj\omega t$ ------(1)

where, $a_j = \frac{2}{\tau}\int_0^{\tau} F(t)cosj\omega tdt,$ j = 0,1,2……..------(2)

and $b_j = \frac{2}{\tau}\int_0^{\tau} F(t)sinj\omega tdt,$ j = 1,2……..------(3)

The equation of motion of the system can be expressed as,

m$\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\propto} a_j cosj\omega t + \sum_{j=1}^{\propto} b_j sinj\omega t$ ------(4)

The right hand side of this equation is a constant plus a sum of harmonic functions. Using the principle of superposition, the steady-state solution of (4) is the sum of the steady-state solution of the following equations:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \text{ ------(5)}$$

$$m\ddot{x} + c\dot{x} + kx = \sum_{j=1}^{\propto} a_j cosj\omega t \text{ -------(6)}$$

$$m\ddot{x} + c\dot{x} + kx = \sum_{j=1}^{\propto} b_j sinj\omega t \text{ -------(7)}$$

The solution of equation 5 is given by;

$$x_p(t) = \frac{a_0}{2} \text{ ------(8)}$$

Also, express the solutions of equation 6 and 7 respectively

$$x_p(t) = \frac{a_j/k}{\sqrt{(1-j^2r^2)^2 + (2\zeta jr)^2}}\cos(j\omega t - \emptyset_j) \text{ ------- (9)}$$

$$x_p(t) = \frac{b_j/k}{\sqrt{(1-j^2r^2)^2 + (2\zeta jr)^2}}\sin(j\omega t - \emptyset_j) \text{ ------- (10)}$$

where, $\emptyset_j = \tan^{-1}(\frac{2\zeta jr}{1-j^2r^2})$------ (11)

and r = $\frac{\omega}{\omega_n}$ ------ (12)

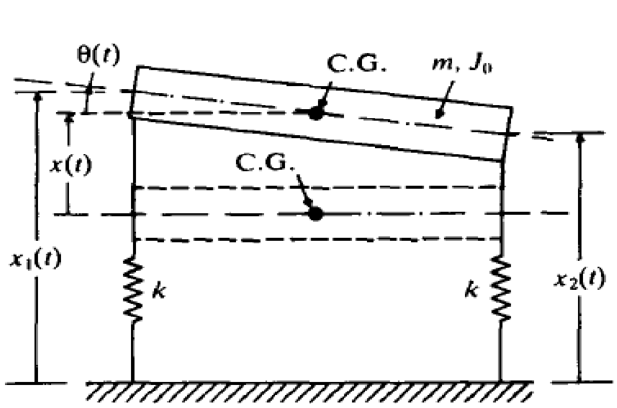
Thus, the complete steady-state solution of 4 is given by,

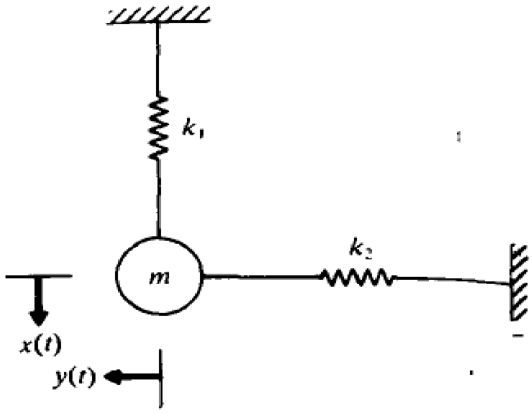
$$x_p(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} \frac{a_j/k}{\sqrt{(1-j^2r^2)^2 + (2\zeta jr)^2}}\cos(j\omega t - \emptyset_j) + \sum_{j=1}^{\infty} \frac{b_j/k}{\sqrt{(1-j^2r^2)^2 + (2\zeta jr)^2}}\sin(j\omega t - \emptyset_j) \text{ -----(13)}$$

Two Degree of Freedom Systems:

Systems that require two independent coordinates to describe their motion are called Two-degree of Freedom systems.

Example:





The general rule for the computation of the number of degrees of freedom can be stated as follows:

Number of degrees of freedom of the system = number of masses in the system X Number of possible types of motion of each mass.

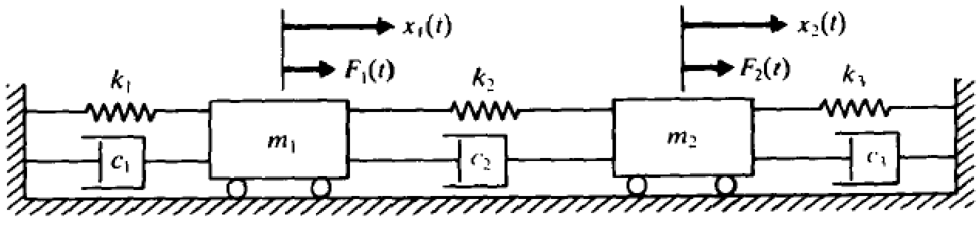
There are two equation of motion for a two degree of freedom system, one for each mass (more precisely, for each degree of freedom).

They are generally in the form of coupled differential equations – i.e., each equation involves all the coordinates. If a harmonic solution is assumed for each coordinate, the equation of motion lead to a frequency equation that gives two natural frequencies for the system. If we give suitable initial excitation, the system vibrates at one of these natural frequencies.

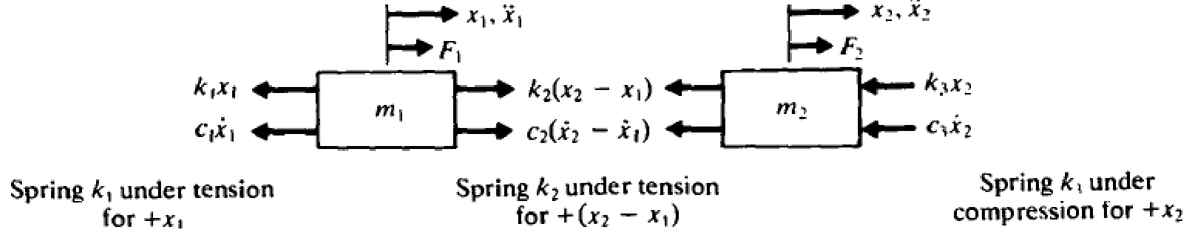
During free vibration at one of the natural frequencies, the amplitudes of the two degrees of freedom (coordinates) are related in a specific manner and the configuration is called a Normal mode, principal mode, or natural mode of vibration. Thus, a two degree of freedom system has two normal modes of vibration corresponding to the two natural frequencies.

Equations of Motion for forced Vibration:

Consider a viscously damped two degree of freedom spring-mass system as shown as figure:



(a)



Now, the motion of the system is completely described by the coordinates$x_1(t) and\ x_2 t$, which define the positions of the masses m_1 and m_2 at any time‘t’ from the respective equilibrium positions. The external forces $F_1(t)$ and $F_2(t)$ act on the masses m_1 and m_2 are shown in figure.

By Newton’s IInd law of motion to each of the masses gives the equation of motion:

$$m_1\ddot{x_1} + (c_1 + c_2)\dot{x_1} - c_2\dot{x_2} + (k_1 + k_2)x_1 - k_2x_2 = F_1 \text{ ------- (1)}$$

$$m_2\ddot{x_2} - c_2\dot{x_1} + (c_2 + c_3)\dot{x_2} - k_2x_1 + (k_2 + k_3)x_2 = F_2 \text{ ------- (2)}$$

equation (1), contains terms involving x_2 [$namely, -c_2\dot{x}_2$ $and -k_2x_2$], and equation (2), contains terms involving x_1 [$namely, -c_2\dot{x}_1$ $and -k_2x_1$],

Hence, they represent a system of two coupled differential equations. We can therefore expect that the motions of the mass m_1 will influence the motion of the mass m_2 and vice-versa.

Equation (1) and (2) can be written in matrix form as;

$$[m]\ddot{\bar{x}}(t) + [c]\dot{\bar{x}}(t) + [k]\bar{x}(t) = \vec{F}(t) \text{ ------ (3)}$$

Where, [m], [c] and [k] are called the mass, damping and stiffness matrices respectively and are given by

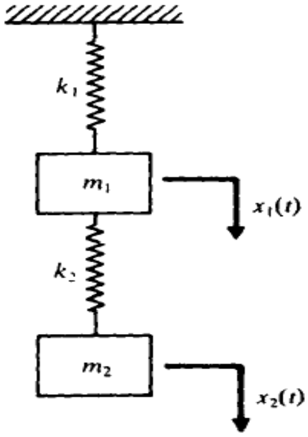
$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, [c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

and $\vec{x}(t) and\ \vec{F}(t)$ are called the displacement and force vectors, respectively and are given by

$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

$$\text{and,} \qquad \vec{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

Problem 1]: Find the natural frequencies of the system as shown in figure with m_1 = m, m_2 = 2m, k_1 = k, and k_2 = 2k. Determine the response of the system when k = 1000 N/m, m 20 kg, and the initial values of the displacements of the masses m_1 and m_2 are 1 and -1 respectively.



Solution: equation of motion,

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0 \text{ ------ (1)}$$
$$m_2\ddot{x}_2 - k_2x_1 + k_2x_2 = 0 \text{ --------- (2)}$$

with $x_i(t) = X_i\cos(\omega t + \emptyset) : i = 1,2.$

equation (1), give the frequency equation,

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

$$\text{or } \omega^4 - \left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)\omega^2 + \frac{k_1k_2}{m_1m_2} = 0 \text{ ------- (3)}$$

roots of equation (3), are

$$\omega_1^2, \omega_2^2 = \frac{k_1+k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}} \text{ -------- (4)}$$

if, $\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1X_1^{(1)} \end{Bmatrix}$ and $\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2X_1^{(2)} \end{Bmatrix}$,

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{k_2}{-\omega_1^2 m_2 + k_2} \text{------- (5)}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{k_2}{-\omega_2^2 m_2 + k_2} \text{------- (6)}$$

General solutions of equation (1) and (2) is

$$x_1(t) = x_1^{(1)} \cos(\omega_1 t + \emptyset_1) + x_1^{(2)} \cos(\omega_2 t + \emptyset_2) \text{ -------- (7)}$$

$$x_2(t) = r_1 x_1^{(1)} \cos(\omega_1 t + \emptyset_1) + r_2 x_1^{(2)} \cos(\omega_2 t + \emptyset_2) \text{ -------- (8)}$$

Where, $x_1^{(1)}, x_1^{(2)}, \emptyset_1, \emptyset_2$can be found using for, $m_1 = m$, $m_2 = 2m$; $k_1 = k$, $k_2 = 2k$, equation (4)

$$\omega_1^2 = (2 - \sqrt{3})\frac{k}{m};\ \omega_2^2 = (2 + \sqrt{3})\frac{k}{m} \text{--------- (9)}$$

When k =1000 N/m, and m = 20 kg, ω_1 = 3.6602 rad/sec, and ω_2 = 13.6603 rad/sec

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = 1.36604, r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = -0.36602$$

with $x_1(0) = 1, \dot{x_1}(0) = 0, x_2(0) = -1, \dot{x_2}(0) = 0,$

gives, $x_1^{(1)} = -0.36602, x_1^{(2)} = -1.36603, \emptyset_1 = 0, \emptyset_2 = 0$

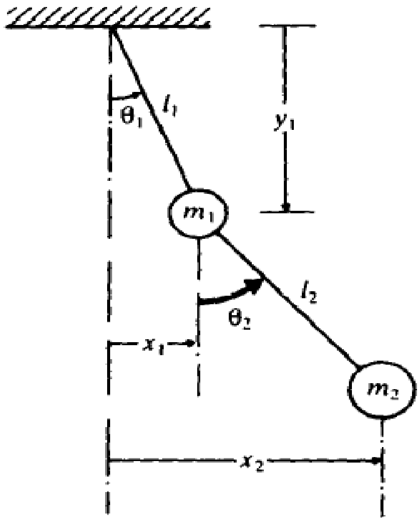
Response of the system is,

$$x_1(t) = -0.36602 \cos(3.6603t) - 1.36603 \cos(13.6603t)$$

$$x_2(t) = -0.5 \cos(3.6603t) + 0.5 \cos(13.6603t)$$

Problem 2]: Set up the differential equations of motion for the double pendulum shown in figure. Using the coordinates x_1 and x_2 and assuming small amplitudes. Find the natural frequencies the ratio of amplitudes, and the locations of nodes for the two modes of vibration when $m_1 = m_2 = m \text{ and } l_1 = l_2 = l.$

Solution:



Taking moment about o and mass m_1,

$m_1 l_1^2 \ddot{\theta_1} = -w_1(l_1 sin\theta_1) + Qsin\theta_2(l_1 cos\theta_1) - Qcos\theta_2(l_1 sin\theta_1) = -w_1 l_1 sin\theta_1 + w_2 l_1(\theta_2 - \theta_1)$--1

Assuming $\theta \cong w_2$,

Similarly,

$$m_2 l_2^2 \ddot{\theta_2} + m_2 l_2 (l_1 \ddot{\theta_1}) = -w_2(l_2 sin\theta_2) = -w_2 l_2 \theta_2 \text{ -------- (2)}$$

using the relations, $\theta_1 = \frac{x_1}{l_1} \text{ and } \theta_2 = \frac{(x_2 - x_1)}{l_2}$

equation (1) and (2) becomes,

$$m_1 l_1^2 \frac{\ddot{x_1}}{l_1} + \left[w_1 + w_2\left(\frac{l_1 + l_2}{l_2}\right)\right] x_1 - \frac{w_2 l_1}{l_2} x_2 = 0 \text{ ------- (3)}$$

and, $m_2 l_2^2 \frac{\ddot{x}_2}{l_2} - w_2 x_1 + w_2 x_2 = 0$ -------- (4)

when, $m_1 = m_2 = m, l_2 = l_1 = l \; and \; w_1 = w_2 = mg$

then equation (3) and (4) becomes,

$$ml\ddot{x}_1 + 3mgx_1 - mgx_2 = 0 \text{ ------ (5)}$$

$$ml\ddot{x}_2 - mgx_1 + mgx_2 = 0 \text{ ------ (6)}$$

For harmonic motion, $x_i(t) = x_i cos\omega t; i = 1, 2, 3, \dots \dots$

Equation (5) and (6) becomes,

$$-\omega^2 mlX_1 + 3mgX_1 - mgX_2 = 0 \text{ ------ (7)}$$

$$-\omega^2 mlX_2 - mgX_1 + mgX_2 = 0 \text{ ------ (8)}$$

From which the frequency equation can be obtained as;

$\omega^4 m^2 l^2 - (4m^2 lg)\omega^2 + 2m^2 g^2 = 0$

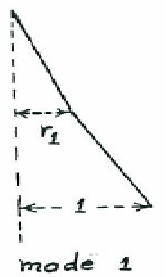
i.e., $\omega_1^2, \omega_2^2 = (2 \pm \sqrt{2})\frac{g}{l}$

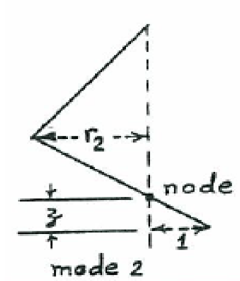
$$\therefore \; \omega_1 = 0.7654\sqrt{\frac{g}{l}}, \omega_2 = 1.8478\sqrt{\frac{g}{l}}$$

Ratio of amplitude is given by,

$$\frac{X_1}{X_2} = \frac{mg}{-\omega^2 ml + 3mg} = \frac{1}{(-\omega^2 \frac{l}{g} + 3)}$$

In mode 1, $\omega_1 = 0.7654\sqrt{\frac{g}{l}}, r_1 = (\frac{X_1}{X_2})^{(1)} = 0.4142$





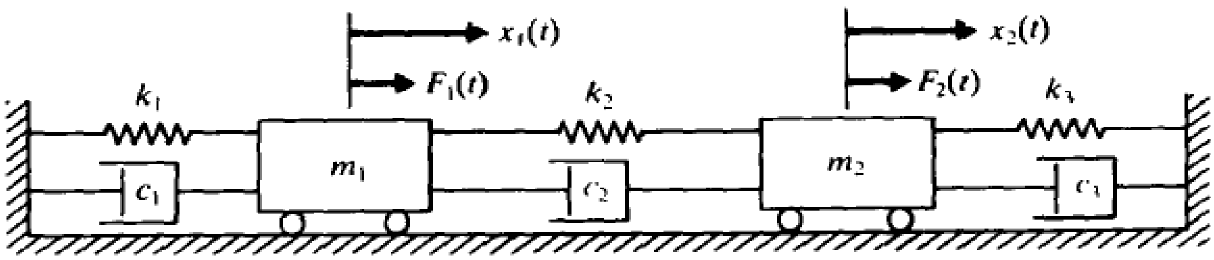
In mode 2, $\omega_2 = 1.8478\sqrt{\frac{g}{l}}, r_2 = (\frac{X_1}{X_2})^{(2)} = -2.4133$

One node located at z:

$$\frac{z}{1} = \frac{1 - z}{2.4133} \; or \; z = 0.2930$$

Free Vibration Analysis of an Undamped System:

For the free vibration analysis of the system shown in figure; we set $f_1(t) = f_2(t) = 0$.



Suppose, if damping is disregarded, $c_1 = c_2 = c_3 = 0$, and the equation of motion,

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2 x_2(t) = 0 \text{ ------ (1)}$$

$$m_2\ddot{x}_2(t) - k_2 x_1(t) + (k_2 + k_3)x_2(t) = 0 \text{ --------- (2)}$$

Here, we are interested in knoeing whether m_1 and $m_2$can oscillate harmonically with the same frequency and phase angle but with different amplitudes.

Assuming that it is possible to have harmonic motion of m_1 and m_2 at the same frequency ω and the same phase angle $\emptyset$, we take the solutions of equation (1) and (2) as

$$x_1(t) = X_1\cos(\omega t + \emptyset) \text{ ------ (3)}$$

$$x_2(t) = X_2\cos(\omega t + \emptyset) \text{ ------ (4)}$$

Where, X_1 and X_2 are constant that denote the maximum amplitudes of $x_1(t)$ and $x_2(t)$, and $\emptyset$ is the phase angle. Substitute equation (3), (4) into equation (1), (2), we obtain,

$$[\{-m_1\omega^2 + (k_1 + k_2)\}X_1 - k_2x_2]\cos(\omega t + \emptyset) = 0 \text{ ------- (5)}$$

$$[\{-m_2\omega^2 + (k_2 + k_3)\}X_2 - k_2x_1]\cos(\omega t + \emptyset) = 0 \text{ ------- (6)}$$

Since equation (5), (6) must be satisfied for all values of the time 't', the terms between brackets must be zero. This yields

$$\{-m_1\omega^2 + (k_1 + k_2)\}X_1 - k_2x_2 = 0 \text{ ------ (7)}$$

$$\{-m_2\omega^2 + (k_2 + k_3)\}X_2 - k_2x_1 = 0 \text{ ------ (8)}$$

Which represent two simultaneously homogeneous algebraic equations in the unknown X_1 and X_2.
It can be seen that equation (7), (8) are satisfied by the trivial solution X_1=X_2 = 0, which implies that there is no vibration.
For a non-trivial solution of X_1 and X_2, the determines of the coefficient of X_1 and X_2 must be zero:

$$\det\begin{bmatrix} \{-m_1\omega^2 + (k_1 + k_2)\} & -k_2 \\ -k_2 & \{-m_2\omega^2 + (k_2 + k_3)\} \end{bmatrix} = 0$$

or, $(m_1m_2)\omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}\omega^2 + \omega\{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0$ ------ (9)

equation (9) is called the frequency or characteristic equation because solution of this equation yields the frequencies or the characteristic values of the system. The roots of equation (9) are given by,

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\} \mp [\left\{\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right\}^2 - 4\{\frac{(k_1+k_2)(k_2+k_3)- k_2^2}{m_1m_2}\}]^{1/2} \text{ ------ (10)}$$

This shoes that it is possible for the system to have a non-trivial harmonic solution of the form of equation (3), (4) when ω is equal to ω_1 or ω_2 gives by equation (10), we call ω_1 and ω_2 the natural frequencies of the system.
The values of X_1 and X_2 remain to be determined. These values depend on the natural frequencies ω_1 and ω_2. We shall denote the values of X_1 and X_2 corresponding to ω_1 as $X_1^{(1)}$ and $X_2^{(1)}$ and those corresponding to ω_2 as$X_1^{(2)}$ and $X_2^{(2)}$. Further, since the equation (7), (8) are homogeneous, only the ratios $r_1 = \{\frac{X_2^{(1)}}{X_1^{(1)}}\}$ and $r_2 = \{\frac{X_2^{(2)}}{X_1^{(2)}}\}$ can be found.
For $\omega^2 = \omega_1^2 = \omega_2^2$, equation (7), (8) gives

$$r_1 = \{\frac{X_2^{(1)}}{X_1^{(1)}}\} = \frac{-m_1\omega_1^2+(k_1+k_2)}{k_2} = \frac{k_2}{-m_2\omega_1^2+(k_2+k_3)} \text{------- (11)}$$

$$r_1 = \{\frac{X_2^{(2)}}{X_1^{(2)}}\} = \frac{-m_1\omega_2^2+(k_1+k_2)}{k_2} = \frac{k_2}{-m_2\omega_2^2+(k_2+k_3)} \text{------- (12)}$$

Torsional System

UNIT IV

MULTI DEGREE OF FREEDOM

value, always smaller than the exact value, of the fundamental natural frequency. Rayleigh's method, which is based on Rayleigh's principle, also gives an approximate value of the fundamental natural frequency, which is always larger than the exact value. Proof is given of Rayleigh's quotient and its stationariness in the neighborhood of an eigenvalue. It is also shown that the Rayleigh's quotient is never lower than the first eigenvalue and never higher than the highest eigenvalue. Use of the static deflection curve in estimating the fundamental natural frequencies of beams and shafts using Rayleigh's method is presented. Holzer's method, based on a trial-and-error scheme, is presented to find the natural frequencies of undamped, damped, semidefinite, or branched translational and torsional systems. The matrix iteration method and its extensions for finding the smallest, highest, and intermediate natural frequencies are presented. A proof for the convergence of the method to the smallest frequency is given. Jacobi's method, which finds all the eigenvalues and eigenvectors of real symmetric matrices, is outlined. The standard eigenvalue problem is defined and the method of deriving it from the general eigenvalue problem, based on the Choleski decomposition method, is presented. Finally, the use of MATLAB in finding the eigenvalues and eigenvectors of multidegree-of-freedom systems is illustrated with several numerical examples.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Find the approximate fundamental frequency of a composite system in terms of the natural frequencies of component parts using Dunkerley's formula.
- Understand Rayleigh's principle, and the properties of Rayleigh's quotient, and compute the fundamental natural frequency of a system using Rayleigh's method.
- Find the approximate natural frequencies of vibration and the modal vectors by using Holzer's method.
- Determine the smallest, intermediate, and highest natural frequencies of a system by using matrix iteration method and its extensions (using matrix deflation procedure).
- Find all the eigenvalues and eigenvectors of a multidegree-of-freedom system using Jacobi's method.
- Convert a general eigenvalue problem into a standard eigenvalue problem based on the Choleski decomposition method.
- Solve eigenvalue problems using MATLAB.

Introduction

In the preceding chapter, the natural frequencies (eigenvalues) and the natural modes (eigenvectors) of a multidegree-of-freedom system were found by setting the characteristic determinant equal to zero. Although this is an exact method, the expansion of the characteristic determinant and the solution of the resulting nth-degree polynomial equation to obtain the natural frequencies can become quite tedious for large values of n. Several analytical and numerical methods have been developed to compute the natural frequencies and mode shapes of multidegree-of-freedom systems. In this chapter, we shall consider Dunkerley's formula, Rayleigh's method, Holzer's method, the matrix iteration method, and Jacobi's method. Dunkerley's formula and Rayleigh's method are useful only for estimating the fundamental natural frequency. Holzer's method is essentially a tabular method that can be used to find partial or full solutions to eigenvalue problems. The matrix iteration method finds one natural frequency at a time, usually starting from the lowest value. The method can thus be terminated after finding the required number of natural frequencies and mode shapes. When all the natural frequencies and mode shapes are required, Jacobi's method can be used; it finds all the eigenvalues and eigenvectors simultaneously.

Dunkerley's Formula

Dunkerley's formula gives the approximate value of the fundamental frequency of a composite system in terms of the natural frequencies of its component parts. It is derived by making use of the fact that the higher natural frequencies of most vibratory systems are large compared to their fundamental frequencies [7.1–7.3]. To derive Dunkerley's formula, consider a general n-degree-of-freedom system whose eigenvalues can be determined by solving the frequency equation, Eq. (6.63):

$$\left| -[k] + \omega^2[m] \right| = 0$$

or

$$\left| -\frac{1}{\omega^2}[I] + [a][m] \right| = 0 \tag{7.1}$$

For a lumped-mass system with a diagonal mass matrix, Eq. (7.1) becomes

$$\left| -\frac{1}{\omega^2}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & m_n \end{bmatrix} \right| = 0$$

—that is,

$$\begin{vmatrix} \left(-\frac{1}{\omega^2} + a_{11}m_1\right) & a_{12}m_2 & \cdots & a_{1n}m_n \\ a_{21}m_1 & \left(-\frac{1}{\omega^2} + a_{22}m_2\right) & \cdots & a_{2n}m_n \\ \vdots & \vdots & & \vdots \\ a_{n1}m_1 & a_{n2}m_2 & \cdots & \left(-\frac{1}{\omega^2} + a_{nn}m_n\right) \end{vmatrix} = 0 \tag{7.2}$$

The expansion of Eq. (7.2) leads to

$$
\begin{aligned}
\left(\frac{1}{\omega^2}\right)^n &- (a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n)\left(\frac{1}{\omega^2}\right)^{n-1} \\
&+ (a_{11}a_{22}m_1m_2 + a_{11}a_{33}m_1m_3 + \cdots + a_{n-1,n-1}a_{nn}m_{n-1}m_n \\
&- a_{12}a_{21}m_1m_2 - \cdots - a_{n-1,n}a_{n,n-1}m_{n-1}m_n)\left(\frac{1}{\omega^2}\right)^{n-2} \\
&- \cdots = 0
\end{aligned}
\tag{7.3}
$$

This is a polynomial equation of *n*th degree in $(1/\omega^2)$. Let the roots of Eq. (7.3) be denoted as $1/\omega_1^2, 1/\omega_2^2, \ldots, 1/\omega_n^2$. Thus

$$
\begin{aligned}
&\left(\frac{1}{\omega^2} - \frac{1}{\omega_1^2}\right)\left(\frac{1}{\omega^2} - \frac{1}{\omega_2^2}\right)\cdots\left(\frac{1}{\omega^2} - \frac{1}{\omega_n^2}\right) \\
&= \left(\frac{1}{\omega^2}\right)^n - \left(\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2}\right)\left(\frac{1}{\omega^2}\right)^{n-1} - \cdots = 0
\end{aligned}
\tag{7.4}
$$

Equating the coefficient of $(1/\omega^2)^{n-1}$ in Eqs. (7.4) and (7.3) gives

$$
\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} + \cdots + \frac{1}{\omega_n^2} = a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \tag{7.5}
$$

In most cases, the higher frequencies $\omega_2, \omega_3, \ldots, \omega_n$ are considerably larger than the fundamental frequency ω_1, and so

$$
\frac{1}{\omega_i^2} \ll \frac{1}{\omega_1^2}, \qquad i = 2, 3, \ldots, n
$$

Thus, Eq. (7.5) can be approximately written as

$$
\frac{1}{\omega_1^2} \simeq a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \tag{7.6}
$$

This equation is known as *Dunkerley's formula*. The fundamental frequency given by Eq. (7.6) will always be smaller than the exact value. In some cases, it will be more convenient to rewrite Eq. (7.6) as

$$
\frac{1}{\omega_1^2} \simeq \frac{1}{\omega_{1n}^2} + \frac{1}{\omega_{2n}^2} + \cdots + \frac{1}{\omega_{nn}^2} \tag{7.7}
$$

where $\omega_{in} = (1/a_{ii}m_i)^{1/2} = (k_{ii}/m_i)^{1/2}$ denotes the natural frequency of a single-degree-of-freedom system consisting of mass m_i and spring of stiffness k_{ii}, $i = 1, 2, \ldots, n$. The use of Dunkerley's formula for finding the lowest frequency of elastic systems is presented in references [7.4, 7.5].

EXAMPLE 7.1

Fundamental Frequency of a Beam

Estimate the fundamental natural frequency of a simply supported beam carrying three identical equally spaced masses, as shown in Fig. 7.1.

Solution: The flexibility influence coefficients (see Example 6.6) required for the application of Dunkerley's formula are given by

$$a_{11} = a_{33} = \frac{3}{256}\frac{l^3}{EI}, \qquad a_{22} = \frac{1}{48}\frac{l^3}{EI} \tag{E.1}$$

Using $m_1 = m_2 = m_3 = m$, Eq. (7.6) thus gives

$$\frac{1}{v_1^2} M + \frac{3}{256} + \frac{1}{48} + \frac{3}{256} * \frac{ml^3}{EI} = 0.04427\frac{ml^3}{EI}$$

$$v_1 \; M \; 4.75375 \frac{\cancel{EI}}{Aml^3}$$

This value can be compared with the exact value of the fundamental frequency, $4.9326 \frac{\cancel{EI}}{Aml^3}$ (see Problem 6.54)

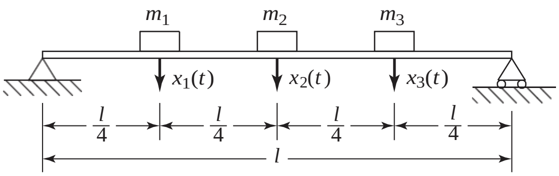


FIGURE 7.1 Beam carrying masses.

*

Rayleigh s Method

Rayleigh's method was presented in Section 2.5 to find the natural frequencies of single-degree-of-freedom systems. The method can be extended to find the approximate value of the fundamental natural frequency of a discrete system.[1] The method is based on *Rayleigh's principle*, which can be stated as follows [7.6]:

> *The frequency of vibration of a conservative system vibrating about an equilibrium position has a stationary value in the neighborhood of a natural mode. This stationary value, in fact, is a minimum value in the neighborhood of the fundamental natural mode.*

We shall now derive an expression for the approximate value of the first natural frequency of a multidegree-of-freedom system according to Rayleigh's method.

[1]Rayleigh's method for continuous systems is presented in Section 8.7

The kinetic and potential energies of an n-degree-of-freedom discrete system can be expressed as

$$T = \frac{1}{2} \dot{\vec{x}}^{T} [m] \dot{\vec{x}} \tag{7.8}$$

$$V = \frac{1}{2} \vec{x}^{T} [k] \vec{x} \tag{7.9}$$

To find the natural frequencies, we assume harmonic motion to be

$$\vec{x} = \vec{X} \cos \omega t \tag{7.10}$$

where $\vec{X}$ denotes the vector of amplitudes (mode shape) and ω represents the natural frequency of vibration. If the system is conservative, the maximum kinetic energy is equal to the maximum potential energy:

$$T_{\max} = V_{\max} \tag{7.11}$$

By substituting Eq. (7.10) into Eqs. (7.8) and (7.9), we find

$$T_{\max} = \frac{1}{2} \vec{X}^{T} [m] \vec{X} \omega^{2} \tag{7.12}$$

$$V_{\max} = \frac{1}{2} \vec{X}^{T} [k] \vec{X} \tag{7.13}$$

By equating $T_{\max}$ and $V_{\max}$, we obtain[2]

$$\omega^{2} = \frac{\vec{X}^{T} [k] \vec{X}}{\vec{X}^{T} [m] \vec{X}} \tag{7.14}$$

The right-hand side of Eq. (7.14) is known as *Rayleigh's quotient* and is denoted as $R(\vec{X})$.

Properties of Rayleigh's Quotient

As stated earlier, $R(\vec{X})$ has a stationary value when the arbitrary vector $\vec{X}$ is in the neighborhood of any eigenvector $\vec{X}^{(r)}$. To prove this, we express the arbitrary vector $\vec{X}$ in terms of the normal modes of the system, $\vec{X}^{(i)}$, as

$$\vec{X} = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)} + c_3 \vec{X}^{(3)} + \cdots \tag{7.15}$$

Then

$$\begin{aligned} \vec{X}^{T} [k] \vec{X} = {} & c_1^2 \vec{X}^{(1)T} [k] \vec{X}^{(1)} + c_2^2 \vec{X}^{(2)T} [k] \vec{X}^{(2)} \\ & + c_3^2 \vec{X}^{(3)T} [k] \vec{X}^{(3)} + \cdots \end{aligned} \tag{7.16}$$

---

[2] Equation (7.14) can also be obtained from the relation $[k]\vec{X} = \omega^{2}[m]\vec{X}$. Premultiplying this equation by $\vec{X}^{T}$ and solving the resulting equation gives Eq. (7.14).

and

$$\vec{X}^T[m]\vec{X} = c_1^2\,\vec{X}^{(1)T}[m]\vec{X}^{(1)} + c_2^2\,\vec{X}^{(2)T}[m]\vec{X}^{(2)} + c_3^2\,\vec{X}^{(3)T}[m]\vec{X}^{(3)} + \cdots \tag{7.17}$$

as the cross terms of the form $c_i c_j \vec{X}^{(i)T}[k]\vec{X}^{(j)}$ and $c_i c_j \vec{X}^{(i)T}[m]\vec{X}^{(j)}$, $i \neq j$, are zero by the orthogonality property. Using Eqs. (7.16) and (7.17) and the relation

$$\vec{X}^{(i)T}[k]\vec{X}^{(i)} = \omega_i^2\,\vec{X}^{(i)T}[m]\vec{X}^{(i)} \tag{7.18}$$

the Rayleigh's quotient of Eq. (7.14) can be expressed as

$$\omega^2 = R(\vec{X}) = \frac{c_1^2\omega_1^2\,\vec{X}^{(1)T}[m]\vec{X}^{(1)} + c_2^2\omega_2^2\,\vec{X}^{(2)T}[m]\vec{X}^{(2)} + \cdots}{c_1^2\,\vec{X}^{(1)T}[m]\vec{X}^{(1)} + c_2^2\,\vec{X}^{(2)T}[m]\vec{X}^{(2)} + \cdots} \tag{7.19}$$

If the normal modes are normalized, this equation becomes

$$\omega^2 = R(\vec{X}) = \frac{c_1^2\omega_1^2 + c_2^2\omega_2^2 + \cdots}{c_1^2 + c_2^2 + \cdots} \tag{7.20}$$

If $\vec{X}$ differs little from the eigenvector $\vec{X}^{(r)}$, the coefficient c_r will be much larger than the remaining coefficients c_i $(i \neq r)$, and Eq. (7.20) can be written as

$$R(\vec{X}) = \frac{c_r^2\omega_r^2 + c_r^2 \displaystyle\sum_{\substack{i=1,2,\ldots\\ i\neq r}} \left(\frac{c_i}{c_r}\right)^2 \omega_i^2}{c_r^2 + c_r^2 \displaystyle\sum_{\substack{i=1,2,\ldots\\ i\neq r}} \left(\frac{c_i}{c_r}\right)^2} \tag{7.21}$$

Since $|c_i/c_r| = \epsilon_i \ll 1$, where ϵ_i is a small number for all $i \neq r$, Eq. (7.21) gives

$$R(\vec{X}) = \omega_r^2\{1 + O(\epsilon^2)\} \tag{7.22}$$

where $O(\epsilon^2)$ represents an expression in ϵ of the second order or higher. Equation (7.22) indicates that if the arbitrary vector $\vec{X}$ differs from the eigenvector $\vec{X}^{(r)}$ by a small quantity of the first order, $R(\vec{X})$ differs from the eigenvalue ω_r^2 by a small quantity of the second order. This means that Rayleigh's quotient has a stationary value in the neighborhood of an eigenvector.

The stationary value is actually a minimum value in the neighborhood of the fundamental mode, $\vec{X}^{(1)}$. To see this, let $r = 1$ in Eq. (7.21) and write

$$R(\vec{X}) = \frac{\omega_1^2 + \sum_{i=2,3,\ldots} \left(\frac{c_i}{c_1}\right)^2 \omega_i^2}{\left\{1 + \sum_{i=2,3,\ldots} \left(\frac{c_i}{c_1}\right)^2\right\}}$$

$$\simeq \omega_1^2 + \sum_{i=2,3,\ldots} \varepsilon_i^2 \omega_i^2 - \omega_1^2 \sum_{i=2,3,\ldots} \varepsilon_i^2$$

$$\simeq \omega_1^2 + \sum_{i=2,3,\ldots} (\omega_i^2 - \omega_1^2)\varepsilon_i^2 \tag{7.23}$$

Since, in general, $\omega_i^2 > \omega_1^2$ for $i = 2, 3, \ldots$, Eq. (7.23) leads to

$$R(\vec{X}) \geq \omega_1^2 \tag{7.24}$$

which shows that Rayleigh's quotient is never lower than the first eigenvalue. By proceeding in a similar manner, we can show that

$$R(\vec{X}) \leq \omega_n^2 \tag{7.25}$$

which means that Rayleigh's quotient is never higher than the highest eigenvalue. Thus Rayleigh's quotient provides an upper bound for ω_1^2 and a lower bound for ω_n^2.

Computation of the Fundamental Natural Frequency

Equation (7.14) can be used to find an approximate value of the first natural frequency (ω_1) of the system. For this, we select a trial vector $\vec{X}$ to represent the first natural mode $\vec{X}^{(1)}$ and substitute it on the right-hand side of Eq. (7.14). This yields the approximate value of ω_1^2. Because Rayleigh's quotient is stationary, remarkably good estimates of ω_1^2 can be obtained even if the trial vector $\vec{X}$ deviates greatly from the true natural mode $\vec{X}^{(1)}$. Obviously, the estimated value of the fundamental frequency ω_1 is more accurate if the trial vector ($\vec{X}$) chosen resembles the true natural mode $\vec{X}^{(1)}$ closely. Rayleigh's method is compared with Dunkerley's and other methods in Refs. [7.7–7.9].

EXAMPLE 7.2

Fundamental Frequency of a Three-Degree-of-Freedom System

Estimate the fundamental frequency of vibration of the system shown in Fig. 7.2. Assume that $m_1 = m_2 = m_3 = m$, $k_1 = k_2 = k_3 = k$, and the mode shape is

$$\vec{X} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

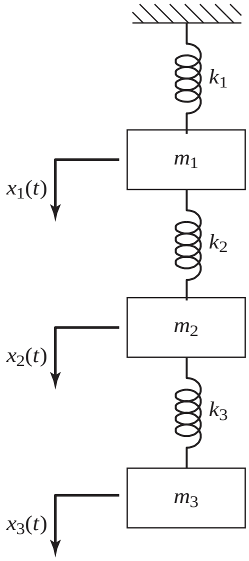


FIGURE 7.2 Three-degree-of-freedom spring-mass system.

Solution: The stiffness and mass matrices of the system are

$$[k] = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \tag{E.1}$$

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{E.2}$$

By substituting the assumed mode shape in the expression for Rayleigh's quotient, we obtain

$$R(\vec{X}) = \omega^2 = \frac{\vec{X}^T[k]\vec{X}}{\vec{X}^T[m]\vec{X}} = \frac{(1\ 2\ 3)k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1\ 2\ 3)m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = 0.2143\frac{k}{m} \tag{E.3}$$

$$\omega_1 = 0.4629\sqrt{\frac{k}{m}} \tag{E.4}$$

This value is 4.0225 percent larger than the exact value of $0.4450\sqrt{k/m}$. The exact fundamental mode shape (see Example 6.10) in this case is

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0000 \\ 1.8019 \\ 2.2470 \end{Bmatrix} \tag{E.5}$$

*

Fundamental Frequency of Beams and Shafts

Although the procedure outlined above is applicable to all discrete systems, a simpler equation can be derived for the fundamental frequency of the lateral vibration of a beam or a shaft carrying several masses such as pulleys, gears, or flywheels. In these cases, the static deflection curve is used as an approximation of the dynamic deflection curve.

Consider a shaft carrying several masses, as shown in Fig. 7.3. The shaft is assumed to have negligible mass. The potential energy of the system is the strain energy of the deflected shaft, which is equal to the work done by the static loads. Thus

$$V_{max} = \frac{1}{2}(m_1 g w_1 + m_2 g w_2 + \cdots) \tag{7.26}$$

where $m_i g$ is the static load due to the mass m_i, and w_i is the total static deflection of mass m_i due to all the masses. For harmonic oscillation (free vibration), the maximum kinetic energy due to the masses is

$$T_{max} = \frac{\omega^2}{2}(m_1 w_1^2 + m_2 w_2^2 + \cdots) \tag{7.27}$$

where ω is the frequency of oscillation. Equating V_{max} and T_{max}, we obtain

$$\omega = \left\{ \frac{g(m_1 w_1 + m_2 w_2 + \cdots)}{(m_1 w_1^2 + m_2 w_2^2 + \cdots)} \right\}^{1/2} \tag{7.28}$$

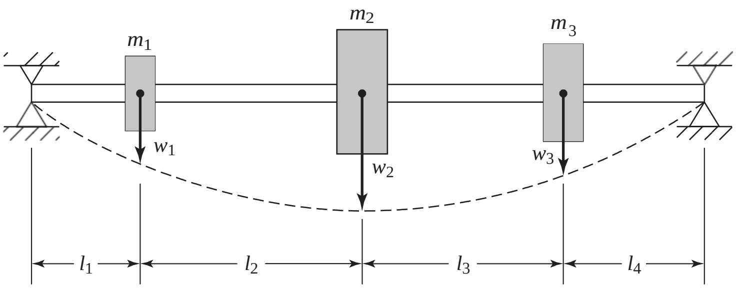


FIGURE 7.3 Shaft carrying masses.

EXAMPLE 7.3 Fundamental Frequency of a Shaft with Rotors

Estimate the fundamental frequency of the lateral vibration of a shaft carrying three rotors (masses), as shown in Fig. 7.3, with $m_1 = 20$ kg, $m_2 = 50$ kg, $m_3 = 40$ kg, $l_1 = 1$ m, $l_2 = 3$ m, $l_3 = 4$ m, and $l_4 = 2$ m. The shaft is made of steel with a solid circular cross section of diameter 10 cm.

Solution: From strength of materials, the deflection of the beam shown in Fig. 7.4 due to a static load P [7.10] is given by

$$w(x) = \begin{cases} \dfrac{Pbx}{6EIl}(l^2 - b^2 - x^2); & 0 \le x \le a \quad \text{(E.1)} \\ -\dfrac{Pa(l - x)}{6EIl}[a^2 + x^2 - 2lx]; & a \le x \le l \quad \text{(E.2)} \end{cases}$$

Deflection Due to the Weight of m_1*:* At the location of mass m_1 (with $x = 1$ m, $b = 9$ m, and $l = 10$ m in Eq. (E.1)):

$$w_1' = \frac{(20 * 9.81)(9)(1)}{6EI(10)}(100 - 81 - 1) = \frac{529.74}{EI} \tag{E.3}$$

At the location of m_2 (with $a = 1$ m, $x = 4$ m, and $l = 10$ m in Eq. (E.2)):

$$w_2' = -\frac{(20 * 9.81)(1)(6)}{6EI(10)}[1 + 16 - 2(10)(4)] = \frac{1236.06}{EI} \tag{E.4}$$

At the location of m_3 (with $a = 1$ m, $x = 8$ m, and $l = 10$ m in Eq. (E.2)):

$$w_3' = -\frac{(20 * 9.81)(1)(2)}{6EI(10)}[1 + 64 - 2(10)(8)] = \frac{621.3}{EI} \tag{E.5}$$

Deflection Due to the Weight of m_2*:* At the location of m_1 (with $x = 1$ m, $b = 6$ m, and $l = 10$ m in Eq. (E.1)):

$$w_1'' = \frac{(50 * 9.81)(6)(1)}{6EI(10)}(100 - 36 - 1) = \frac{3090.15}{EI} \tag{E.6}$$

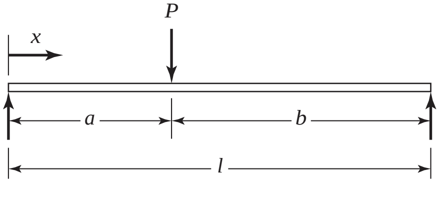


FIGURE 7.4 Beam under static load.

At the location of m_2 (with $x = 4$ m, $b = 6$ m, and $l = 10$ m in Eq. (E.1)):

$$w_2'' = \frac{(50 \times 9.81)(6)(4)}{6EI(10)}(100 - 36 - 16) = \frac{9417.6}{EI} \tag{E.7}$$

At the location of m_3 (with $a = 4$ m, $x = 8$ m, and $l = 10$ m in Eq. (E.2)):

$$w_3'' = -\frac{(50 \times 9.81)(4)(2)}{6EI(10)}[16 + 64 - 2(10)(8)] = \frac{5232.0}{EI} \tag{E.8}$$

Deflection Due to the Weight of m_3*:* At the location of m_1 (with $x = 1$ m, $b = 2$ m, and $l = 10$ m in Eq. (E.1)):

$$w_1''' = \frac{(40 \times 9.81)(2)(1)}{6EI(10)}(100 - 4 - 1) = \frac{1242.6}{EI} \tag{E.9}$$

At the location of m_2 (with $x = 4$ m, $b = 2$ m, and $l = 10$ m in Eq. (E.1)):

$$w_2''' = \frac{(40 \times 9.81)(2)(4)}{6EI(10)}(100 - 4 - 16) = \frac{4185.6}{EI} \tag{E.10}$$

At the location of m_3 (with $x = 8$ m, $b = 2$ m, and $l = 10$ m in Eq. (E.1)):

$$w_3''' = \frac{(40 \times 9.81)(2)(8)}{6EI(10)}(100 - 4 - 64) = \frac{3348.48}{EI} \tag{E.11}$$

The total deflections of the masses m_1, m_2, and m_3 are

$$w_1 = w_1' + w_1'' + w_1''' = \frac{4862.49}{EI}$$

$$w_2 = w_2' + w_2'' + w_2''' = \frac{14839.26}{EI}$$

$$w_3 = w_3' + w_3'' + w_3''' = \frac{9201.78}{EI}$$

Substituting into Eq. (7.28), we find the fundamental natural frequency:

$$\omega = \left(\frac{9.81(20 \times 4862.49 + 50 \times 14839.26 + 40 \times 9201.78)EI}{20 \times (4862.49)^2 + 50 \times (14839.26)^2 + 40 \times (9201.78)^2} \right)^{1/2}$$

$$= 0.028222\sqrt{EI} \tag{E.12}$$

For the shaft, $E = 2.07 \times 10^{11}$ N/m^2 and $I = \pi(0.1)^4/64 = 4.90875 \times 10^{-6}$ m^4 and hence Eq. (E.12) gives

$$\omega = 28.4482 \text{ rad/s}$$

Holzer's Method

Holzer's method is essentially a trial-and-error scheme to find the natural frequencies of undamped, damped, semidefinite, fixed, or branched vibrating systems involving linear and angular displacements [7.11, 7.12]. The method can also be programmed for computer applications. A trial frequency of the system is first assumed, and a solution is found when the assumed frequency satisfies the constraints of the system. This generally requires several trials. Depending on the trial frequency used, the fundamental as well as the higher frequencies of the system can be determined. The method also gives the mode shapes.

7.4.1 Torsional Systems

Consider the undamped torsional semidefinite system shown in Fig. 7.5. The equations of motion of the discs can be derived as follows:

$$J_1\ddot{\theta}_1 + k_{t1}(\theta_1 - \theta_2) = 0 \tag{7.29}$$

$$J_2\ddot{\theta}_2 + k_{t1}(\theta_2 - \theta_1) + k_{t2}(\theta_2 - \theta_3) = 0 \tag{7.30}$$

$$J_3\ddot{\theta}_3 + k_{t2}(\theta_3 - \theta_2) = 0 \tag{7.31}$$

Since the motion is harmonic in a natural mode of vibration, we assume that $\theta_i = \Theta_i \cos(\omega t + \phi)$ in Eqs. (7.29) to (7.31) and obtain

$$\omega^2 J_1\Theta_1 = k_{t1}(\Theta_1 - \Theta_2) \tag{7.32}$$

$$\omega^2 J_2\Theta_2 = k_{t1}(\Theta_2 - \Theta_1) + k_{t2}(\Theta_2 - \Theta_3) \tag{7.33}$$

$$\omega^2 J_3\Theta_3 = k_{t2}(\Theta_3 - \Theta_2) \tag{7.34}$$

Summing these equations gives

$$\sum_{i=1}^{3} \omega^2 J_i\Theta_i = 0 \tag{7.35}$$

Equation (7.35) states that the sum of the inertia torques of the semidefinite system must be zero. This equation can be treated as another form of the frequency equation, and the trial frequency must satisfy this requirement.

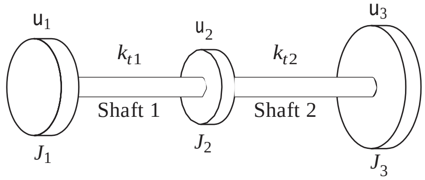


FIGURE 7.5 Torsional semidefinite system.

In Holzer's method, a trial frequency ω is assumed, and θ_1 is arbitrarily chosen as unity. Next, θ_2 is computed from Eq. (7.32), and then θ_3 is found from Eq. (7.33). Thus we obtain

$$\theta_1 = 1 \tag{7.36}$$

$$\theta_2 = \theta_1 - \frac{\omega^2 J_1 \theta_1}{k_{t1}} \tag{7.37}$$

$$\theta_3 = \theta_2 - \frac{\omega^2}{k_{t2}}(J_1\theta_1 + J_2\theta_2) \tag{7.38}$$

These values are substituted in Eq. (7.35) to verify whether the constraint is satisfied. If Eq. (7.35) is not satisfied, a new trial value of ω is assumed and the process repeated. Equations (7.35), (7.37), and (7.38) can be generalized for an *n*-disc system as follows:

$$\sum_{i=1}^{n} \omega^2 J_i \theta_i = 0 \tag{7.39}$$

$$\theta_i = \theta_{i-1} - \frac{\omega^2}{k_{t\,i-1}}\left(\sum_{k=1}^{i-1} J_k \theta_k\right), \qquad i = 2, 3, \ldots, n \tag{7.40}$$

Thus the method uses Eqs. (7.39) and (7.40) repeatedly for different trial frequencies. If the assumed trial frequency is not a natural frequency of the system, Eq. (7.39) is not satisfied. The resultant torque in Eq. (7.39) represents a torque applied at the last disc. This torque M_t is then plotted for the chosen ω. When the calculation is repeated with other values of ω, the resulting graph appears as shown in Fig. 7.6. From this graph, the natural frequencies of the system can be identified as the values of ω at which $M_t = 0$. The amplitudes θ_i ($i = 1, 2, \ldots, n$) corresponding to the natural frequencies are the mode shapes of the system.

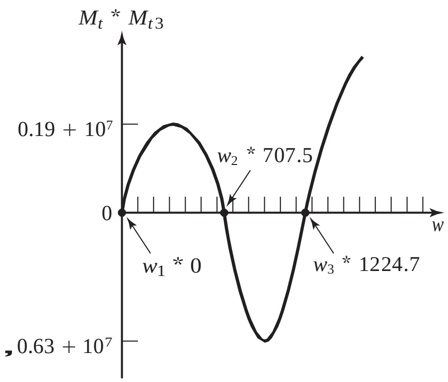


FIGURE 7.6 Resultant torque versus frequency.

Holzer's method can also be applied to systems with fixed ends. At a fixed end, the amplitude of vibration must be zero. In this case, the natural frequencies can be found by plotting the resulting amplitude (instead of the resultant torque) against the assumed frequencies. For a system with one end free and the other end fixed, Eq. (7.40) can be used for checking the amplitude at the fixed end. An improvement of Holzer's method is presented in references [7.13, 7.14].

EXAMPLE 7.4 Natural Frequencies of a Torsional System

The arrangement of the compressor, turbine, and generator in a thermal power plant is shown in Fig. 7.7. Find the natural frequencies and mode shapes of the system.

Solution: This system represents an unrestrained or free-free torsional system. Table 7.1 shows its parameters and the sequence of computations. The calculations for the trial frequencies v = 0, 10, 20, 700, and 710 are shown in this table. The quantity M_{t3} denotes the torque to the right of Station 3

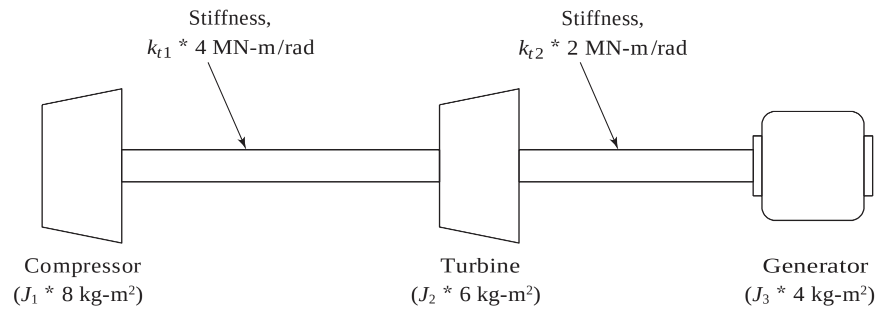


FIGURE 7.7 Free-free torsional system.

TABLE 7.1

Parameters of the System	Quantity	Trial 1	2	3	Á	71	72
		0	10	20		700	710
	v^2	0	100	400		490000	504100
Station 1:							
$J_1 = 8$	$®_1$	1.0	1.0	1.0		1.0	1.0
$k_{t1} = 4 * 10^6$	$M_{t1} = v^2 J_1 ®_1$	0	800	3200		0.392*E*7	0.403*E*7
Station 2:							
$J_2 = 6$	$®_2 = 1 - \frac{M_{t1}}{k_{t1}}$	1.0	0.9998	0.9992		0.0200	−0.0082
$k_{t2} = 2 * 10^6$	$M_{t2} = M_{t1} + v^2 J_2 ®_2$	0	1400	5598		0.398*E*7	0.401*E*7
Station 3:							
$J_3 = 4$	$®_3 = ®_2 - \frac{M_{t2}}{k_{t2}}$	1.0	0.9991	0.9964		−1.9690	−2.0120
$K_{t3} = 0$	$M_{t3} = M_{t2} + v^2 J_3 ®_3$	0	1800	7192		0.119*E*6	−0.494E5

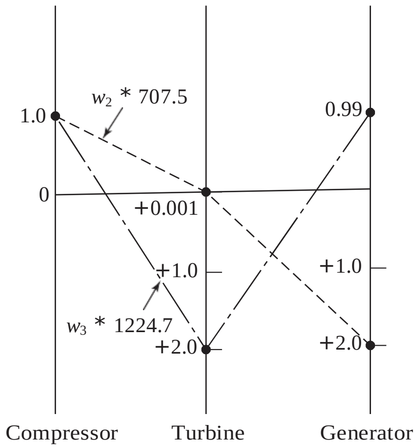


FIGURE 7.8 First two flexible modes.

(generator), which must be zero at the natural frequencies. Figure 7.6 shows the graph of M_{t3} versus ω. Closely spaced trial values of ω are used in the vicinity of $M_{t3} = 0$ to obtain accurate values of the first two flexible mode shapes, shown in Fig. 7.8. Note that the value $\omega = 0$ corresponds to the rigid-body rotation.

*

7.4.2 Spring-Mass Systems

Although Holzer's method has been extensively applied to torsional systems, the procedure is equally applicable to the vibration analysis of spring-mass systems. The equations of motion of a spring-mass system (see Fig. 7.9) can be expressed as

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = 0 \tag{7.41}$$

$$m_2\ddot{x}_2 + k_1(x_2 - x_1) + k_2(x_2 - x_3) = 0$$
$$\vdots \tag{7.42}$$

For harmonic motion, $x_i(t) = X_i \cos \omega t$, where X_i is the amplitude of mass m_i, and Eqs. (7.41) and (7.42) can be written as

$$\omega^2 m_1 X_1 = k_1(X_1 - X_2) \tag{7.43}$$

$$\omega^2 m_2 X_2 = k_1(X_2 - X_1) + k_2(X_2 - X_3)$$
$$= -\omega^2 m_1 X_1 + k_2(X_2 - X_3) \tag{7.44}$$
$$\vdots$$

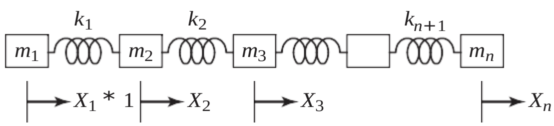


FIGURE 7.9 Free-free spring mass system.

The procedure for Holzer's method starts with a trial frequency ω and the amplitude of mass m_1 as $X_1 = 1$. Equations (7.43) and (7.44) can then be used to obtain the amplitudes of the masses $m_2, m_3, \ldots, m_i$:

$$X_2 = X_1 - \frac{\omega^2 m_1 X_1}{k_1} \tag{7.45}$$

$$X_3 = X_2 - \frac{\omega^2}{k_2}(m_1 X_1 + m_2 X_2) \tag{7.46}$$

$$X_i = X_{i-1} - \frac{\omega^2}{k_{i-1}}\left(\sum_{k=1}^{i-1} m_k X_k\right), \qquad i = 2, 3, \ldots, n \tag{7.47}$$

As in the case of torsional systems, the resultant force applied to the last (nth) mass can be computed as follows:

$$F = \sum_{i=1}^{n} \omega^2 m_i X_i \tag{7.48}$$

The calculations are repeated with several other trial frequencies ω. The natural frequencies are identified as those values of ω that give $F = 0$ for a free-free system. For this, it is convenient to plot a graph between F and ω, using the same procedure for spring-mass systems as for torsional systems.

Matrix Iteration Method

The matrix iteration method assumes that the natural frequencies are distinct and well separated such that $\omega_1 < \omega_2 < \cdots < \omega_n$. The iteration is started by selecting a trial vector $\vec{X}_1$, which is then premultiplied by the dynamical matrix $[D]$. The resulting column vector is then normalized, usually by making one of its components equal to unity. The normalized column vector is premultiplied by $[D]$ to obtain a third column vector, which is normalized in the same way as before and becomes still another trial column vector. The process is repeated until the successive normalized column vectors converge to a common vector: the fundamental eigenvector. The normalizing factor gives the largest value of $\lambda = 1/\omega^2$—that is, the smallest or the fundamental natural frequency [7.15]. The convergence of the process can be explained as follows.

According to the expansion theorem, any arbitrary n-dimensional vector $\vec{X}_1$ can be expressed as a linear combination of the n orthogonal eigenvectors of the system $\vec{X}^{(i)}$, $i = 1, 2, \ldots, n$:

$$\vec{X}_1 = c_1 \vec{X}^{(1)} + c_2 \vec{X}^{(2)} + \cdots + c_n \vec{X}^{(n)} \tag{7.49}$$

where $c_1, c_2, \ldots, c_n$ are constants. In the iteration method, the trial vector $\vec{X}_1$ is selected arbitrarily and is therefore a known vector. The modal vectors $\vec{X}^{(i)}$, although unknown, are constant vectors because they depend upon the properties of the system. The constants c_i

are unknown numbers to be determined. According to the iteration method, we premultiply $\vec{X}_1$ by the matrix $[D]$. In view of Eq. (7.49), this gives

$$[D]\vec{X}_1 = c_1[D]\vec{X}^{(1)} + c_2[D]\vec{X}^{(2)} + \cdots + c_n[D]\vec{X}^{(n)} \tag{7.50}$$

Now, according to Eq. (6.66), we have

$$[D]\vec{X}^{(i)} = \lambda_i[I]\vec{X}^{(i)} = \frac{1}{\omega_i^2}\vec{X}^{(i)}; \qquad i = 1, 2, \ldots, n \tag{7.51}$$

Substitution of Eq. (7.51) into Eq. (7.50) yields

$$\begin{aligned}[D]\vec{X}_1 &= \vec{X}_2 \\ &= \frac{c_1}{\omega_1^2}\vec{X}^{(1)} + \frac{c_2}{\omega_2^2}\vec{X}^{(2)} + \cdots + \frac{c_n}{\omega_n^2}\vec{X}^{(n)}\end{aligned} \tag{7.52}$$

where $\vec{X}_2$ is the second trial vector. We now repeat the process and premultiply $\vec{X}_2$ by $[D]$ to obtain, by Eqs. (7.49) and (6.66),

$$\begin{aligned}[D]\vec{X}_2 &= \vec{X}_3 \\ &= \frac{c_1}{\omega_1^4}\vec{X}^{(1)} + \frac{c_2}{\omega_2^4}\vec{X}^{(2)} + \cdots + \frac{c_n}{\omega_n^4}\vec{X}^{(n)}\end{aligned} \tag{7.53}$$

By repeating the process we obtain, after the rth iteration,

$$\begin{aligned}[D]\vec{X}_r &= \vec{X}_{r+1} \\ &= \frac{c_1}{\omega_1^{2r}}\vec{X}^{(1)} + \frac{c_2}{\omega_2^{2r}}\vec{X}^{(2)} + \cdots + \frac{c_n}{\omega_n^{2r}}\vec{X}^{(n)}\end{aligned} \tag{7.54}$$

Since the natural frequencies are assumed to be $\omega_1 < \omega_2 < \cdots < \omega_n$, a sufficiently large value of r yields

$$\frac{1}{\omega_1^{2r}} \gg \frac{1}{\omega_2^{2r}} \gg \cdots \gg \frac{1}{\omega_n^{2r}} \tag{7.55}$$

Thus the first term on the right-hand side of Eq. (7.54) becomes the only significant one. Hence we have

$$\vec{X}_{r+1} = \frac{c_1}{\omega_1^{2r}}\vec{X}^{(1)} \tag{7.56}$$

which means that the $(r + 1)$th trial vector becomes identical to the fundamental modal vector to within a multiplicative constant. Since

$$\vec{X}_r = \frac{c_1}{\omega_1^{2(r-1)}}\vec{X}^{(1)} \tag{7.57}$$

the fundamental natural frequency ω_1 can be found by taking the ratio of any two corresponding components in the vectors $\vec{X}_r$ and $\vec{X}_{r+1}$:

$$\omega_1^2 \simeq \frac{X_{i,r}}{X_{i,r+1}}, \qquad \text{for any } i = 1, 2, \ldots, n \tag{7.58}$$

where $X_{i,r}$ and $X_{i,r+1}$ are the *i*th elements of the vectors $\vec{X}_r$ and $\vec{X}_{r+1}$, respectively.

Discussion

1. In the above proof, nothing has been said about the normalization of the successive trial vectors $\vec{X}_i$. Actually, it is not necessary to establish the proof of convergence of the method. The normalization amounts to a readjustment of the constants $c_1, c_2, \ldots, c_n$ in each iteration.
2. Although it is theoretically necessary to have $r \to \infty$ for the convergence of the method, in practice only a finite number of iterations suffices to obtain a reasonably good estimate of ω_1.
3. The actual number of iterations necessary to find the value of ω_1 to within a desired degree of accuracy depends on how closely the arbitrary trial vector $\vec{X}_1$ resembles the fundamental mode $\vec{X}^{(1)}$ and on how well ω_1 and ω_2 are separated. The required number of iterations is less if ω_2 is very large compared to ω_1.
4. The method has a distinct advantage in that any computational errors made do not yield incorrect results. Any error made in premultiplying $\vec{X}_i$ by $[D]$ results in a vector other than the desired one, $\vec{X}_{i+1}$. But this wrong vector can be considered as a new trial vector. This may delay the convergence but does not produce wrong results.
5. One can take any set of *n* numbers for the first trial vector $\vec{X}_1$ and still achieve convergence to the fundamental modal vector. Only in the unusual case in which the trial vector $\vec{X}_1$ is exactly proportional to one of the modes $\vec{X}^{(i)}$ $(i \neq 1)$ does the method fail to converge to the first mode. In such a case, the premultiplication of $\vec{X}^{(i)}$ by $[D]$ results in a vector proportional to $\vec{X}^{(i)}$ itself.

7.5.1 Convergence to the Highest Natural Frequency

To obtain the highest natural frequency ω_n and the corresponding mode shape or eigenvector $\vec{X}^{(n)}$ by the matrix iteration method, we first rewrite Eq. (6.66) as

$$[D]^{-1}\vec{X} = \omega^2[I]\vec{X} = \omega^2\vec{X} \tag{7.59}$$

where $[D]^{-1}$ is the inverse of the dynamical matrix $[D]$ given by

$$[D]^{-1} = [m]^{-1}[k] \tag{7.60}$$

Now we select any arbitrary trial vector $\vec{X}_1$ and premultiply it by $[D]^{-1}$ to obtain an improved trial vector $\vec{X}_2$. The sequence of trial vectors $\vec{X}_{i+1}$ $(i = 1, 2, \ldots)$ obtained by premultiplying by $[D]^{-1}$ converges to the highest normal mode $\vec{X}^{(n)}$. It can be seen that the procedure is similar to the one already described. The constant of proportionality in this case is ω^2 instead of $1/\omega^2$.

7.5.2 Computation of Intermediate Natural Frequencies

Once the first natural frequency ω_1 (or the largest eigenvalue $\lambda_1 = 1/\omega_1^2$) and the corresponding eigenvector $\vec{X}^{(1)}$ are determined, we can proceed to find the higher natural frequencies and the corresponding mode shapes by the matrix iteration method. Before we proceed, it should be remembered that any arbitrary trial vector premultiplied by $[D]$ would lead again to the largest eigenvalue. It is thus necessary to remove the largest eigenvalue from the matrix $[D]$. The succeeding eigenvalues and eigenvectors can be obtained by eliminating the root λ_1 from the characteristic or frequency equation

$$\left| [D] - \lambda[I] \right| = 0 \tag{7.61}$$

A procedure known as *matrix deflation* can be used for this purpose [7.16]. To find the eigenvector $\vec{X}^{(i)}$ by this procedure, the previous eigenvector $\vec{X}^{(i-1)}$ is normalized with respect to the mass matrix such that

$$\vec{X}^{(i-1)T}[m]\vec{X}^{(i-1)} = 1 \tag{7.62}$$

The deflated matrix $[D_i]$ is then constructed as

$$[D_i] = [D_{i-1}] - \lambda_{i-1}\vec{X}^{(i-1)}\vec{X}^{(i-1)T}[m], \qquad i = 2, 3, \ldots, n \tag{7.63}$$

where $[D_1] = [D]$. Once $[D_i]$ is constructed, the iterative scheme

$$\vec{X}_{r+1} = [D_i]\vec{X}_r \tag{7.64}$$

is used, where $\vec{X}_1$ is an arbitrary trial eigenvector.

EXAMPLE 7.5 Natural Frequencies of a Three-Degree-of-Freedom System

Find the natural frequencies and mode shapes of the system shown in Fig. 7.2 for $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m_3 = m$ by the matrix iteration method.

Solution: The mass and stiffness matrices of the system are given in Example 7.2. The flexibility matrix is

$$[a] = [k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{E.1}$$

and so the dynamical matrix is

$$[k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{E.2}$$

The eigenvalue problem can be stated as

$$[D]\vec{X} = \lambda\vec{X} \tag{E.3}$$

where

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \tag{E.4}$$

and

$$\lambda = \frac{k}{m} \cdot \frac{1}{\omega^2} \tag{E.5}$$

First Natural Frequency: By assuming the first trial eigenvector or mode shape to be

$$\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{E.6}$$

the second trial eigenvector can be obtained:

$$\vec{X}_2 = [D]\vec{X}_1 = \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} \tag{E.7}$$

By making the first element equal to unity, we obtain

$$\vec{X}_2 = 3.0 \begin{Bmatrix} 1.0000 \\ 1.6667 \\ 2.0000 \end{Bmatrix} \tag{E.8}$$

and the corresponding eigenvalue is given by

$$\lambda_1 \simeq 3.0 \qquad \text{or} \qquad \omega_1 \simeq 0.5773\sqrt{\frac{k}{m}} \tag{E.9}$$

The subsequent trial eigenvector can be obtained from the relation

$$\vec{X}_{i+1} = [D]\vec{X}_i \tag{E.10}$$

and the corresponding eigenvalues are given by

$$\lambda_1 \simeq X_{1,\,i+1} \tag{E.11}$$

where $X_{1,i+1}$ is the first component of the vector $\vec{X}_{i+1}$ before normalization. The various trial eigenvectors and eigenvalues obtained by using Eqs. (E.10) and (E.11) are shown in the table below.

i	$\vec{X}_i$ with $X_{1,i} = 1$	$\vec{X}_{i+1} = [D]\vec{X}_i$	$\lambda_1 \simeq X_{1,i+1}$	ω_1
1	$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix}$	3.0	$0.5773\sqrt{\frac{k}{m}}$
2	$\begin{Bmatrix} 1.00000 \\ 1.66667 \\ 2.00000 \end{Bmatrix}$	$\begin{Bmatrix} 4.66667 \\ 8.33333 \\ 10.33333 \end{Bmatrix}$	4.66667	$0.4629\sqrt{\frac{k}{m}}$
3	$\begin{Bmatrix} 1.0000 \\ 1.7857 \\ 2.2143 \end{Bmatrix}$	$\begin{Bmatrix} 5.00000 \\ 9.00000 \\ 11.2143 \end{Bmatrix}$	5.00000	$0.4472\sqrt{\frac{k}{m}}$
⋮				
7	$\begin{Bmatrix} 1.00000 \\ 1.80193 \\ 2.24697 \end{Bmatrix}$	$\begin{Bmatrix} 5.04891 \\ 9.09781 \\ 11.34478 \end{Bmatrix}$	5.04891	$0.44504\sqrt{\frac{k}{m}}$
8	$\begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$	$\begin{Bmatrix} 5.04892 \\ 9.09783 \\ 11.34481 \end{Bmatrix}$	5.04892	$0.44504\sqrt{\frac{k}{m}}$

It can be seen that the mode shape and the natural frequency converged (to the fourth decimal place) in eight iterations. Thus the first eigenvalue and the corresponding natural frequency and mode shape are given by

$$\lambda_1 = 5.04892, \qquad \omega_1 = 0.44504\sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix} \tag{E.12}$$

Second Natural Frequency: To compute the second eigenvalue and the eigenvector, we must first produce a deflated matrix:

$$[D_2] = [D_1] - \lambda_1 \vec{X}^{(1)} \vec{X}^{(1)T}[m] \tag{E.13}$$

This equation, however, calls for a normalized vector $\vec{X}^{(1)}$ satisfying $\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1$. Let the normalized vector be denoted as

$$\vec{X}^{(1)} = \alpha \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

where α is a constant whose value must be such that

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = \alpha^2 m \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.00000 \\ 1.80194 \\ 2.24698 \end{Bmatrix}$$

$$= \alpha^2 m(9.29591) = 1 \tag{E.14}$$

from which we obtain $\alpha = 0.32799m^{-1/2}$. Hence the first normalized eigenvector is

$$\vec{X}^{(1)} = m^{-1/2} \begin{Bmatrix} 0.32799 \\ 0.59102 \\ 0.73699 \end{Bmatrix} \tag{E.15}$$

Next we use Eq. (E.13) and form the first deflated matrix:

$$[D_2] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} - 5.04892 \begin{Bmatrix} 0.32799 \\ 0.59102 \\ 0.73699 \end{Bmatrix} \begin{Bmatrix} 0.32799 \\ 0.59102 \\ 0.73699 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.45684 & 0.02127 & -0.22048 \\ 0.02127 & 0.23641 & -0.19921 \\ -0.22048 & -0.19921 & 0.25768 \end{bmatrix} \tag{E.16}$$

Since the trial vector can be chosen arbitrarily, we again take

$$\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \tag{E.17}$$

By using the iterative scheme

$$\vec{X}_{i+1} = [D_2]\vec{X}_i \tag{E.18}$$

we obtain $\vec{X}_2$

$$\vec{X}_2 = \begin{Bmatrix} 0.25763 \\ 0.05847 \\ -0.16201 \end{Bmatrix} = 0.25763 \begin{Bmatrix} 1.00000 \\ 0.22695 \\ -0.62885 \end{Bmatrix} \tag{E.19}$$

Hence λ_2 can be found from the general relation

$$\lambda_2 = X_{1,i+1} \tag{E.20}$$

as 0.25763. Continuation of this procedure gives the results shown in the table below.

i	$\vec{X}_i$ with $X_{1,i} = 1$	$\vec{X}_{i+1} = [D_2]\vec{X}_i$	$\lambda_2 = X_{1,i+1}$	ω_2
1	$\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 0.25763 \\ 0.05847 \\ -0.16201 \end{Bmatrix}$	0.25763	$1.97016\sqrt{\frac{k}{m}}$
2	$\begin{Bmatrix} 1.00000 \\ 0.22695 \\ -0.62885 \end{Bmatrix}$	$\begin{Bmatrix} 0.60032 \\ 0.20020 \\ -0.42773 \end{Bmatrix}$	0.60032	$1.29065\sqrt{\frac{k}{m}}$
.				
.				
.				
10	$\begin{Bmatrix} 1.00000 \\ 0.44443 \\ -0.80149 \end{Bmatrix}$	$\begin{Bmatrix} 0.64300 \\ 0.28600 \\ -0.51554 \end{Bmatrix}$	0.64300	$1.24708\sqrt{\frac{k}{m}}$
11	$\begin{Bmatrix} 1.00000 \\ 0.44479 \\ -0.80177 \end{Bmatrix}$	$\begin{Bmatrix} 0.64307 \\ 0.28614 \\ -0.51569 \end{Bmatrix}$	0.64307	$1.24701\sqrt{\frac{k}{m}}$

Thus the converged second eigenvalue and the eigenvector are

$$\lambda_2 = 0.64307, \qquad \omega_2 = 1.24701\sqrt{\frac{k}{m}}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} 1.00000 \\ 0.44496 \\ -0.80192 \end{Bmatrix} \tag{E.21}$$

Third Natural Frequency: For the third eigenvalue and the eigenvector, we use a similar procedure. The detailed calculations are left as an exercise to the reader. Note that before computing the deflated matrix $[D_3]$, we need to normalize $\vec{X}^{(2)}$ by using Eq. (7.62), which gives

$$\vec{X}^{(2)} = m^{-1/2}\begin{Bmatrix} 0.73700 \\ 0.32794 \\ -0.59102 \end{Bmatrix} \tag{E.22}$$

*

Jacobi's Method

The matrix iteration method described in the preceding section produces the eigenvalues and eigenvectors of matrix $[D]$ one at a time. Jacobi's method is also an iterative method but produces all the eigenvalues and eigenvectors of $[D]$ simultaneously, where $[D] = [d_{ij}]$ is a real symmetric matrix of order $n \times n$. The method is based on a theorem in linear algebra stating that a real symmetric matrix $[D]$ has only real eigenvalues and that there exists a real orthogonal matrix $[R]$ such that $[R]^T[D][R]$ is diagonal [7.17]. The diagonal elements are the eigenvalues, and the columns of the matrix $[R]$ are the eigenvectors. According to Jacobi's method, the matrix $[R]$ is generated as a product of several rotation matrices [7.18] of the form

$$\underset{n \times n}{[R_1]} = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \\ \left[\begin{array}{ccccccc} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & \ddots & & & & \\ & & & \cos\theta & \ddots & -\sin\theta & \\ & & & & \ddots & & \\ & & & \sin\theta & & \cos\theta & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \text{ith row} \\ \\ \text{jth row} \\ \\ \\ \end{array} \qquad (7.65)$$

$$\begin{array}{cc} \text{ith column} & \text{jth column} \end{array}$$

where all elements other than those appearing in columns and rows i and j are identical with those of the identity matrix $[I]$. If the sine and cosine entries appear in positions (i, i), (i, j), (j, i), and (j, j), then the corresponding elements of $[R_1]^T[D][R_1]$ can be computed as follows:

$$\underline{d}_{ii} = d_{ii}\cos^2\theta + 2d_{ij}\sin\theta\cos\theta + d_{jj}\sin^2\theta \qquad (7.66)$$

$$\underline{d}_{ij} = \underline{d}_{ji} = (d_{jj} - d_{ii})\sin\theta\cos\theta + d_{ij}(\cos^2\theta - \sin^2\theta) \qquad (7.67)$$

$$\underline{d}_{jj} = d_{ii}\sin^2\theta - 2d_{ij}\sin\theta\cos\theta + d_{jj}\cos^2\theta \qquad (7.68)$$

If θ is chosen to be

$$\tan 2\theta = \left(\frac{2d_{ij}}{d_{ii} - d_{jj}}\right) \qquad (7.69)$$

then it makes $\underline{d}_{ij} = \underline{d}_{ji} = 0$. Thus each step of Jacobi's method reduces a pair of off-diagonal elements to zero. Unfortunately, in the next step, while the method reduces a new pair of zeros, it introduces nonzero contributions to formerly zero positions. However, successive matrices of the form

$$[R_2]^T[R_1]^T[D][R_1][R_2], \qquad [R_3]^T[R_2]^T[R_1]^T[D][R_1][R_2][R_3], \ldots$$

converge to the required diagonal form; the final matrix $[R]$, whose columns give the eigenvectors, then becomes

$$[R] = [R_1][R_2][R_3]\cdots \tag{7.70}$$

EXAMPLE 7.6

Eigenvalue Solution Using Jacobi Method

Find the eigenvalues and eigenvectors of the matrix

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

using Jacobi's method.

Solution: We start with the largest off-diagonal term, $d_{23} = 2$, in the matrix $[D]$ and try to reduce it to zero. From Eq. (7.69),

$$\theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{2d_{23}}{d_{22} - d_{33}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{4}{2-3}\right) = -37.981878^\circ$$

$$[R_1] = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.7882054 & 0.6154122 \\ 0.0 & -0.6154122 & 0.7882054 \end{bmatrix}$$

$$[D'] = [R_1]^T[D][R_1] = \begin{bmatrix} 1.0 & 0.1727932 & 1.4036176 \\ 0.1727932 & 0.4384472 & 0.0 \\ 1.4036176 & 0.0 & 4.5615525 \end{bmatrix}$$

Next we try to reduce the largest off-diagonal term of $[D']$—namely, $d'_{13} = 1.4036176$—to zero. Equation (7.69) gives

$$\theta_2 = \frac{1}{2}\tan^{-1}\left(\frac{2d'_{13}}{d'_{11} - d'_{33}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2.8072352}{1.0 - 4.5615525}\right) = -19.122686^\circ$$

$$[R_2] = \begin{bmatrix} 0.9448193 & 0.0 & 0.3275920 \\ 0.0 & 1.0 & 0.0 \\ -0.3275920 & 0.0 & 0.9448193 \end{bmatrix}$$

$$[D''] = [R_2]^T[D'][R_2] = \begin{bmatrix} 0.5133313 & 0.1632584 & 0.0 \\ 0.1632584 & 0.4384472 & 0.0566057 \\ 0.0 & 0.0566057 & 5.0482211 \end{bmatrix}$$

The largest off-diagonal element in $[D'']$ is $d''_{12} = 0.1632584$. θ_3 can be obtained from Eq. (7.69) as

$$\theta_3 = \frac{1}{2}\tan^{-1}\left(\frac{2d''_{12}}{d''_{11} - d''_{22}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{0.3265167}{0.5133313 - 0.4384472}\right) = 38.541515°$$

$$[R_3] = \begin{bmatrix} 0.7821569 & -0.6230815 & 0.0 \\ 0.6230815 & 0.7821569 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

$$[D'''] = [R_3]^T[D''][R_3] = \begin{bmatrix} 0.6433861 & 0.0 & 0.0352699 \\ 0.0 & 0.3083924 & 0.0442745 \\ 0.0352699 & 0.0442745 & 5.0482211 \end{bmatrix}$$

Assuming that all the off-diagonal terms in $[D''']$ are close to zero, we can stop the process here. The diagonal elements of $[D''']$ give the eigenvalues (values of $1/\omega^2$) as 0.6433861, 0.3083924, and 5.0482211. The corresponding eigenvectors are given by the columns of the matrix $[R]$, where

$$[R] = [R_1][R_2][R_3] = \begin{bmatrix} 0.7389969 & -0.5886994 & 0.3275920 \\ 0.3334301 & 0.7421160 & 0.5814533 \\ -0.5854125 & -0.3204631 & 0.7447116 \end{bmatrix}$$

The iterative process can be continued for obtaining a more accurate solution. The present eigenvalues can be compared with the exact values: 0.6431041, 0.3079786, and 5.0489173.

*

Standard Eigenvalue Problem

In the preceding chapter, the eigenvalue problem was stated as

$$[k]\vec{X} = \omega^2[m]\vec{X} \tag{7.71}$$

which can be rewritten in the form of a standard eigenvalue problem [7.19] as

$$[D]\vec{X} = \lambda\vec{X} \tag{7.72}$$

where

$$[D] = [k]^{-1}[m] \tag{7.73}$$

and

$$\lambda = \frac{1}{\omega^2} \tag{7.74}$$

UNIT V
CONTINOUS SYSTEM

element of the particular system and applying the Newton's second law of motion. The free-vibration solution of the system is found by assuming harmonic motion and applying the relevant boundary conditions. The solution gives infinite number of natural frequencies and the corresponding mode shapes. The free-vibration displacement of the system is found as a linear superposition of the mode shapes, the constants involved being determined from the known initial conditions of the system. In the case of transverse vibration of a string of infinite length, the traveling-wave solution is presented. In the case of the longitudinal vibration of a bar, the vibration response under an initial force is also found. In the case of the transverse vibration of beams, all the common boundary conditions are summarized and the orthogonality of normal modes is proved. The forced vibration of beams is presented using the mode superposition method. The effect of axial force on the natural frequencies and mode shapes of beams is considered. The *thick beam theory*, also called the *Timoshenko beam theory*, is presented by considering the effects of rotary inertia and shear deformation. The free vibration of rectangular membranes is presented. *Rayleigh's method*, based on Rayleigh's quotient, for finding the approximate fundamental frequencies of continuous systems is outlined. The extension of the method, known as the *Rayleigh-Ritz* method, is outlined for determining approximate values of several frequencies. Finally, MATLAB solutions are presented for the free and forced vibration of typical continuous systems.

Learning Objectives

After you have finished studying this chapter, you should be able to do the following:

- Derive the equation of motion of a continuous system from the free-body diagram of an infinitesimally small element of the system and Newton's second law.
- Find the natural frequencies and mode shapes of the system using harmonic solution.
- Determine the free-vibration solution using a linear superposition of the mode shapes and the initial conditions.
- Find the free-vibration solutions of string, bar, shaft, beam, and membrane problems.
- Express the vibration of an infinite string in the form of traveling waves.
- Determine the forced-vibration solution of continuous systems using mode superposition method.
- Find the effects of axial force, rotary inertia, and shear deformation on the vibration of beams.
- Apply the Rayleigh and Rayleigh-Ritz methods to find the approximate natural frequencies of continuous systems.
- Use MATLAB to find the natural frequencies, mode shapes, and forced response of continuous systems.

Introduction

We have so far dealt with discrete systems where mass, damping, and elasticity were assumed to be present only at certain discrete points in the system. In many cases, known as *distributed* or *continuous systems*, it is not possible to identify discrete masses, dampers, or springs. We must then consider the continuous distribution of the mass, damping, and elasticity and assume that each of the infinite number of points of the system can vibrate. This is why a continuous system is also called a *system of infinite degrees of freedom.*

If a system is modeled as a discrete one, the governing equations are ordinary differential equations, which are relatively easy to solve. On the other hand, if the system is modeled as a continuous one, the governing equations are partial differential equations, which are more difficult. However, the information obtained from a discrete model of a system may not be as accurate as that obtained from a continuous model. The choice between the two models must be made carefully, with due consideration of factors such as the purpose of the analysis, the influence of the analysis on design, and the computational time available.

In this chapter, we shall consider the vibration of simple continuous systems — strings, bars, shafts, beams, and membranes. A more specialized treatment of the vibration of continuous structural elements is given in references [8.1–8.3]. In general, the frequency equation of a continuous system is a transcendental equation that yields an infinite number of natural frequencies and normal modes. This is in contrast to the behavior of discrete systems, which yield a finite number of such frequencies and modes. We need to apply boundary conditions to find the natural frequencies of a continuous system. The question of boundary conditions does not arise in the case of discrete systems except in an indirect way, because the influence coefficients depend on the manner in which the system is supported.

8.2 Transverse Vibration of a String or Cable

8.2.1 Equation of Motion

Consider a tightly stretched elastic string or cable of length l subjected to a transverse force $f(x, t)$ per unit length, as shown in Fig. 8.1(a). The transverse displacement of the string, $w(x, t)$, is assumed to be small. Equilibrium of the forces in the z direction gives (see Fig. 8.1(b)):

The net force acting on an element is equal to the inertia force acting on the element, or

$$(P + dP)\sin(\theta + d\theta) + f\,dx - P\sin\theta = \rho\,dx\,\frac{\partial^2 w}{\partial t^2} \tag{8.1}$$

where P is the tension, ρ is the mass per unit length, and θ is the angle the deflected string makes with the x-axis. For an elemental length dx,

$$dP = \frac{\partial P}{\partial x}\,dx \tag{8.2}$$

$$\sin\theta \simeq \tan\theta = \frac{\partial w}{\partial x} \tag{8.3}$$

and

$$\sin(\theta + d\theta) \simeq \tan(\theta + d\theta) = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\,dx \tag{8.4}$$

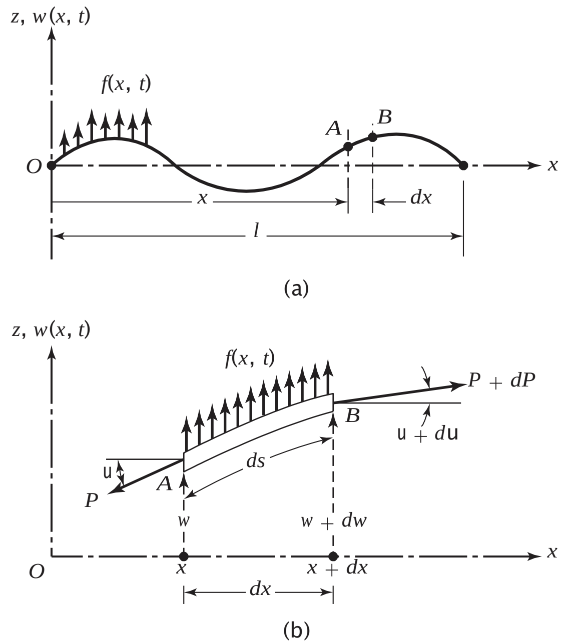


FIGURE 8.1 A vibrating string.

Hence the forced-vibration equation of the nonuniform string, Eq. (8.1), can be simplified to

$$\frac{\partial}{\partial x}\left[P\frac{\partial w(x,t)}{\partial x}\right] + f(x,t) = \rho(x)\frac{\partial^2 w(x,t)}{\partial t^2} \tag{8.5}$$

If the string is uniform and the tension is constant, Eq. (8.5) reduces to

$$P\frac{\partial^2 w(x,t)}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 w(x,t)}{\partial t^2} \tag{8.6}$$

If $f(x,t) = 0$, we obtain the free-vibration equation

$$P\frac{\partial^2 w(x,t)}{\partial x^2} = \rho\frac{\partial^2 w(x,t)}{\partial t^2} \tag{8.7}$$

or

$$c^2\frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2} \tag{8.8}$$

where

$$c = \left(\frac{P}{\rho}\right)^{1/2} \tag{8.9}$$

Equation (8.8) is also known as the *wave equation.*

8.2.2 Initial and Boundary Conditions

The equation of motion, Eq. (8.5) or its special forms (8.6) and (8.7), is a partial differential equation of the second order. Since the order of the highest derivative of w with respect to x and t in this equation is two, we need to specify two boundary and two initial conditions in finding the solution $w(x, t)$. If the string has a known deflection $w_0(x)$ and velocity $\dot{w}_0(x)$ at time $t = 0$, the initial conditions are specified as

$$\begin{aligned} w(x, t = 0) &= w_0(x) \\ \frac{\partial w}{\partial t}(x, t = 0) &= \dot{w}_0(x) \end{aligned} \tag{8.10}$$

If the string is fixed at an end, say $x = 0$, the displacement w must always be zero, and so the boundary condition is

$$w(x = 0, t) = 0, \qquad t \geq 0 \tag{8.11}$$

If the string or cable is connected to a pin that can move in a perpendicular direction as shown in Fig. 8.2, the end cannot support a transverse force. Hence the boundary condition becomes

$$P(x)\frac{\partial w(x, t)}{\partial x} = 0 \tag{8.12}$$

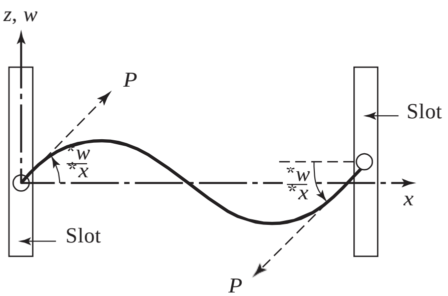


FIGURE 8.2 String connected to pins at the ends.

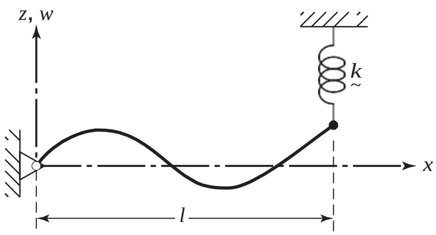


FIGURE 8.3 String with elastic constraint.

If the end $x = 0$ is free and P is a constant, then Eq. (8.12) becomes

$$\frac{\partial w(0, t)}{\partial x} = 0, \qquad t \geq 0 \tag{8.13}$$

If the end $x = l$ is constrained elastically as shown in Fig. 8.3, the boundary condition becomes

$$P(x) \left. \frac{\partial w(x, t)}{\partial x} \right|_{x=l} = -k\, w(x, t)\big|_{x=l}, \qquad t \geq 0 \tag{8.14}$$

where k is the spring constant.

8.2.3 Free Vibration of a Uniform String

The free-vibration equation, Eq. (8.8), can be solved by the method of separation of variables. In this method, the solution is written as the product of a function $W(x)$ (which depends only on x) and a function $T(t)$ (which depends only on t) [8.4]:

$$w(x, t) = W(x)T(t) \tag{8.15}$$

Substitution of Eq. (8.15) into Eq. (8.8) leads to

$$\frac{c^2}{W}\frac{d^2W}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2} \tag{8.16}$$

Since the left-hand side of this equation depends only on x and the right-hand side depends only on t, their common value must be a constant—say, a—so that

$$\frac{c^2}{W}\frac{d^2W}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2} = a \tag{8.17}$$

The equations implied in Eq. (8.17) can be written as

$$\frac{d^2W}{dx^2} - \frac{a}{c^2}W = 0 \tag{8.18}$$

$$\frac{d^2T}{dt^2} - aT = 0 \tag{8.19}$$

Since the constant a is generally negative (see Problem 8.9), we can set $a = -\omega^2$ and write Eqs. (8.18) and (8.19) as

$$\frac{d^2W}{dx^2} + \frac{\omega^2}{c^2} W = 0 \tag{8.20}$$

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \tag{8.21}$$

The solutions of these equations are given by

$$W(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c} \tag{8.22}$$

$$T(t) = C \cos \omega t + D \sin \omega t \tag{8.23}$$

where ω is the frequency of vibration and the constants A, B, C, and D can be evaluated from the boundary and initial conditions.

8.2.4 Free Vibration of a String with Both Ends Fixed

If the string is fixed at both ends, the boundary conditions are $w(0, t) = w(l, t) = 0$ for all time $t \geq 0$. Hence, from Eq. (8.15), we obtain

$$W(0) = 0 \tag{8.24}$$

$$W(l) = 0 \tag{8.25}$$

In order to satisfy Eq. (8.24), A must be zero in Eq. (8.22). Equation (8.25) requires that

$$B \sin \frac{\omega l}{c} = 0 \tag{8.26}$$

Since B cannot be zero for a nontrivial solution, we have

$$\sin \frac{\omega l}{c} = 0 \tag{8.27}$$

Equation (8.27) is called the *frequency* or *characteristic equation* and is satisfied by several values of ω. The values of ω are called the *eigenvalues* (or *natural frequencies* or *characteristic values*) of the problem. The nth natural frequency is given by

$$\frac{\omega_n l}{c} = n\pi, \qquad n = 1, 2, \ldots$$

or

$$\omega_n = \frac{nc\pi}{l}, \qquad n = 1, 2, \ldots \tag{8.28}$$

The solution $w_n(x, t)$ corresponding to v_n can be expressed as

$$w_n(x, t) = W_n(x)T_n(t) = \sin\frac{npx}{l} B C_n \cos\frac{ncpt}{l} + D_n \sin\frac{ncpt}{l} R \tag{8.29}$$

where C_n and D_n are arbitrary constants. The solution $w_n(x, t)$ is called the *nth mode of vibration* or *nth harmonic* or *nth normal mode* of the string. In this mode, each point of the string vibrates with an amplitude proportional to the value of W_n at that point, with the circular frequency $v_n = (ncp)/l$. The function $W_n(x)$ is called the *n*th normal mode, or characteristic function. The first three modes of vibration are shown in Fig. 8.4. The mode corresponding to $n = 1$ is called the *fundamental mode*, and v_1 is called the *fundamental frequency*. The fundamental period is

$$T_1 = \frac{2p}{v_1} = \frac{2l}{c}$$

The points at which $w_n = 0$ for all times are called *nodes*. Thus the fundamental mode has two nodes, at $x = 0$ and $x = l$; the second mode has three nodes, at $x = 0$, $x = l/2$, and $x = l$; etc.

The general solution of Eq. (8.8), which satisfies the boundary conditions of Eqs. (8.24) and (8.25), is given by the superposition of all $w_n(x, t)$:

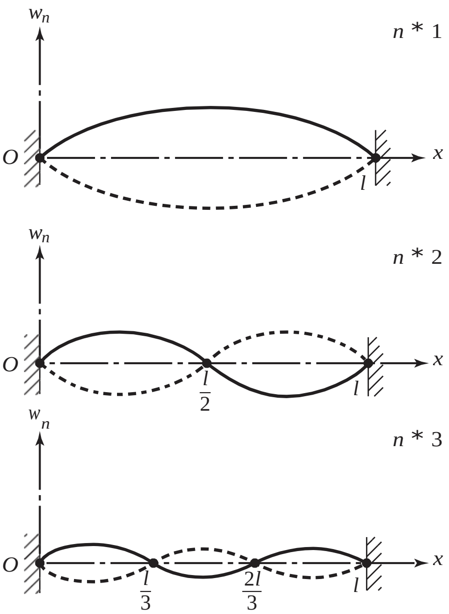


FIGURE 8.4 Mode shapes of a string.

$$w(x, t) = \sum_{n=1}^{\infty} w_n(x, t)$$

$$= \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[C_n \cos\frac{nc\pi t}{l} + D_n \sin\frac{nc\pi t}{l}\right] \tag{8.30}$$

This equation gives all possible vibrations of the string; the particular vibration that occurs is uniquely determined by the specified initial conditions. The initial conditions give unique values of the constants C_n and D_n. If the initial conditions are specified as in Eq. (8.10), we obtain

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} = w_0(x) \tag{8.31}$$

$$\sum_{n=1}^{\infty} \frac{nc\pi}{l} D_n \sin\frac{n\pi x}{l} = \dot{w}_0(x) \tag{8.32}$$

which can be seen to be Fourier sine series expansions of $w_0(x)$ and $\dot{w}_0(x)$ in the interval $0 \le x \le l$. The values of C_n and D_n can be determined by multiplying Eqs. (8.31) and (8.32) by $\sin(n\pi x/l)$ and integrating with respect to x from 0 to l:

$$C_n = \frac{2}{l}\int_0^l w_0(x) \sin\frac{n\pi x}{l}\,dx \tag{8.33}$$

$$D_n = \frac{2}{nc\pi}\int_0^l \dot{w}_0(x) \sin\frac{n\pi x}{l}\,dx \tag{8.34}$$

Note: The solution given by Eq. (8.30) can be identified as the *mode superposition method* since the response is expressed as a superposition of the normal modes. The procedure is applicable in finding not only the free-vibration solution but also the forced-vibration solution of continuous systems.

EXAMPLE 8.1 Dynamic Response of a Plucked String

If a string of length l, fixed at both ends, is plucked at its midpoint as shown in Fig. 8.5 and then released, determine its subsequent motion.

Solution: The solution is given by Eq. (8.30) with C_n and D_n given by Eqs. (8.33) and (8.34), respectively. Since there is no initial velocity, $\dot{w}_0(x) = 0$, and so $D_n = 0$. Thus the solution of Eq. (8.30) reduces to

$$w(x, t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \cos\frac{nc\pi t}{l} \tag{E.1}$$

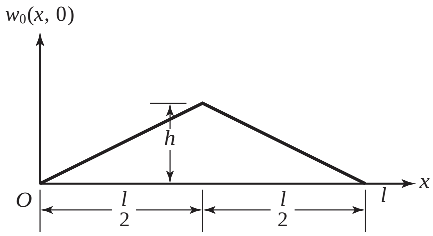


FIGURE 8.5 Initial deflection of the string.

where

$$C_n = \frac{2}{l}\int_0^l w_0(x) \sin\frac{n\pi x}{l}\,dx \tag{E.2}$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} \dfrac{2hx}{l} & \text{for } 0 \le x \le \dfrac{l}{2} \\ \dfrac{2h(l-x)}{l} & \text{for } \dfrac{l}{2} \le x \le l \end{cases} \tag{E.3}$$

By substituting Eq. (E.3) into Eq. (E.2), C_n can be evaluated:

$$C_n = \frac{2}{l}\left\{\int_0^{l/2} \frac{2hx}{l}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2h}{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right\}$$

$$= \begin{cases} \dfrac{8h}{\pi^2 n^2}\sin\dfrac{n\pi}{2} & \text{for } n = 1, 3, 5, \ldots \\ 0 & \text{for } n = 2, 4, 6, \ldots \end{cases} \tag{E.4}$$

By using the relation

$$\sin\frac{n\pi}{2} = (-1)^{(n-1)/2}, \qquad n = 1, 3, 5, \ldots \tag{E.5}$$

the desired solution can be expressed as

$$w(x, t) = \frac{8h}{\pi^2}\left\{\sin\frac{\pi x}{l}\cos\frac{\pi c t}{l} - \frac{1}{9}\sin\frac{3\pi x}{l}\cos\frac{3\pi c t}{l} + \cdots\right\} \tag{E.6}$$

In this case, no even harmonics are excited.

*

8.2.5 Traveling-Wave Solution

The solution of the wave equation, Eq. (8.8), for a string of infinite length can be expressed as [8.5]

$$w(x, t) = w_1(x - ct) + w_2(x + ct) \tag{8.35}$$

where w_1 and w_2 are arbitrary functions of $(x - ct)$ and $(x + ct)$, respectively. To show that Eq. (8.35) is the correct solution of Eq. (8.8), we first differentiate Eq. (8.35):

$$\frac{\partial^2 w(x, t)}{\partial x^2} = w_1''(x - ct) + w_2''(x + ct) \tag{8.36}$$

$$\frac{\partial^2 w(x, t)}{\partial t^2} = c^2 w_1''(x - ct) + c^2 w_2''(x + ct) \tag{8.37}$$

Substitution of these equations into Eq. (8.8) reveals that the wave equation is satisfied. In Eq. (8.35), $w_1(x - ct)$ and $w_2(x + ct)$ represent waves that propagate in the positive and negative directions of the x-axis, respectively, with a velocity c.

For a given problem, the arbitrary functions w_1 and w_2 are determined from the initial conditions, Eq. (8.10). Substitution of Eq. (8.35) into Eq. (8.10) gives, at $t = 0$,

$$w_1(x) + w_2(x) = w_0(x) \tag{8.38}$$

$$-c w_1'(x) + c w_2'(x) = \dot{w}_0(x) \tag{8.39}$$

where the prime indicates differentiation with respect to the respective argument at $t = 0$ (that is, with respect to x). Integration of Eq. (8.39) yields

$$-w_1(x) + w_2(x) = \frac{1}{c}\int_{x_0}^{x} \dot{w}_0(x')\, dx' \tag{8.40}$$

where x_0 is a constant. Solution of Eqs. (8.38) and (8.40) gives w_1 and w_2:

$$w_1(x) = \frac{1}{2}\left[w_0(x) - \frac{1}{c}\int_{x_0}^{x} \dot{w}_0(x')\, dx'\right] \tag{8.41}$$

$$w_2(x) = \frac{1}{2}\left[w_0(x) + \frac{1}{c}\int_{x_0}^{x} \dot{w}_0(x')\, dx'\right] \tag{8.42}$$

By replacing x by $(x - ct)$ and $(x + ct)$, respectively, in Eqs. (8.41) and (8.42), we obtain the total solution:

$$\begin{aligned} w(x, t) &= w_1(x - ct) + w_2(x + ct) \\ &= \frac{1}{2}[w_0(x - ct) + w_0(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \dot{w}_0(x')\, dx' \end{aligned} \tag{8.43}$$

The following points should be noted:

1. As can be seen from Eq. (8.43), there is no need to apply boundary conditions to the problem.
2. The solution given by Eq. (8.43) can be expressed as

$$w(x, t) = w_D(x, t) + w_V(x, t) \tag{8.44}$$

where $w_D(x, t)$ denotes the waves propagating due to the known initial displacement $w_0(x)$ with zero initial velocity, and $w_V(x, t)$ represents waves traveling due only to the known initial velocity $\dot{w}_0(x)$ with zero initial displacement.

The transverse vibration of a string fixed at both ends excited by the transverse impact of an elastic load at an intermediate point was considered in [8.6]. A review of the literature on the dynamics of cables and chains was given by Triantafyllou [8.7].

Longitudinal Vibration of a Bar or Rod

8.3.1 Equation of Motion and Solution

Consider an elastic bar of length l with varying cross-sectional area $A(x)$, as shown in Fig. 8.6. The forces acting on the cross sections of a small element of the bar are given by P and $P + dP$ with

$$P = \sigma A = EA \frac{\partial u}{\partial x} \tag{8.45}$$

where σ is the axial stress, E is Young's modulus, u is the axial displacement, and $\partial u / \partial x$ is the axial strain. If $f(x, t)$ denotes the external force per unit length, the summation of the forces in the x direction gives the equation of motion

$$(P + dP) + f\,dx - P = \rho A\, dx \frac{\partial^2 u}{\partial t^2} \tag{8.46}$$

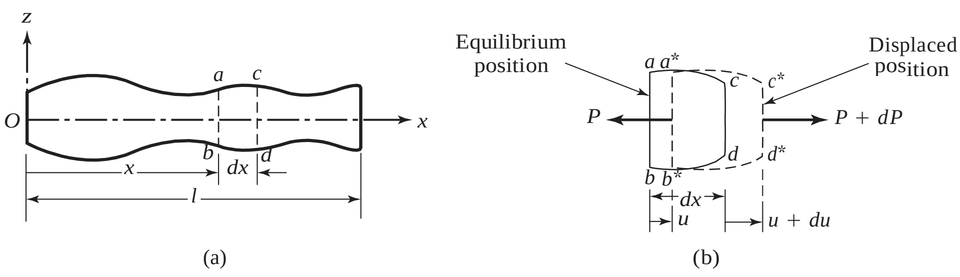


FIGURE 8.6 Longitudinal vibration of a bar.

where ρ is the mass density of the bar. By using the relation $dP = (\partial P/\partial x)\,dx$ and Eq. (8.45), the equation of motion for the forced longitudinal vibration of a nonuniform bar, Eq. (8.46), can be expressed as

$$\frac{\partial}{\partial x}\left[EA(x)\frac{\partial u(x,t)}{\partial x}\right] + f(x,t) = \rho(x)A(x)\frac{\partial^2 u}{\partial t^2}(x,t) \tag{8.47}$$

For a uniform bar, Eq. (8.47) reduces to

$$EA\frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t) = \rho A\frac{\partial^2 u}{\partial t^2}(x,t) \tag{8.48}$$

The free-vibration equation can be obtained from Eq. (8.48), by setting $f = 0$, as

$$c^2\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 u}{\partial t^2}(x,t) \tag{8.49}$$

where

$$c = \sqrt{\frac{E}{\rho}} \tag{8.50}$$

Note that Eqs. (8.47) to (8.50) can be seen to be similar to Eqs. (8.5), (8.6), (8.8), and (8.9), respectively. The solution of Eq. (8.49), which can be obtained as in the case of Eq. (8.8), can thus be written as

$$u(x,t) = U(x)T(t) \equiv \left(\mathcal{A}\cos\frac{\omega x}{c} + \mathcal{B}\sin\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t)^{1} \tag{8.51}$$

End Conditions of Bar	Boundary Conditions	Frequency Equation	Mode Shape (Normal Function)	Natural Frequencies
Fixed-free	$u(0,t) = 0$ $\frac{\partial u}{\partial x}(l,t) = 0$	$\cos\frac{\omega l}{c} = 0$	$U_n(x) = C_n \sin\frac{(2n+1)\pi x}{2l}$	$\omega_n = \frac{(2n+1)\pi c}{2l}$; $n = 0, 1, 2, \ldots$
Free-free	$\frac{\partial u}{\partial x}(0,t) = 0$ $\frac{\partial u}{\partial x}(l,t) = 0$	$\sin\frac{\omega l}{c} = 0$	$U_n(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$; $n = 0, 1, 2, \ldots$
Fixed-fixed	$u(0,t) = 0$ $u(l,t) = 0$	$\sin\frac{\omega l}{c} = 0$	$U_n(x) = C_n \cos\frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l}$; $n = 1, 2, 3, \ldots$

FIGURE 8.7 Common boundary conditions for a bar in longitudinal vibration.

[1]We use $\mathcal{A}$ and $\mathcal{B}$ in this section; A is used to denote the cross-sectional area of the bar.

where the function $U(x)$ represents the normal mode and depends only on x and the function $T(t)$ depends only on t. If the bar has known initial axial displacement $u_0(x)$ and initial velocity $\dot{u}_0(x)$, the initial conditions can be stated as

$$u(x, t = 0) = u_0(x)$$
$$\frac{\partial u}{\partial t}(x, t = 0) = \dot{u}_0(x) \tag{8.52}$$

The common boundary conditions and the corresponding frequency equations for the longitudinal vibration of uniform bars are shown in Fig. 8.7.

EXAMPLE 8.2 Boundary Conditions for a Bar

A uniform bar of cross-sectional area A, length l, and Young's modulus E is connected at both ends by springs, dampers, and masses, as shown in Fig. 8.8(a). State the boundary conditions.

Solution: The free-body diagrams of the masses m_1 and m_2 are shown in Fig. 8.8(b). From this, we find that at the left end $(x = 0)$, the force developed in the bar due to positive u and $\partial u/\partial x$ must be equal to the sum of spring, damper, and inertia forces:

$$AE\frac{\partial u}{\partial x}(0, t) = k_1 u(0, t) + c_1\frac{\partial u}{\partial t}(0, t) + m_1\frac{\partial^2 u}{\partial t^2}(0, t) \tag{E.1}$$

Similarly at the right end $(x = l)$, the force developed in the bar due to positive u and $\partial u/\partial x$ must be equal to the negative sum of spring, damper, and inertia forces:

$$AE\frac{\partial u}{\partial x}(l, t) = -k_2 u(l, t) - c_2\frac{\partial u}{\partial t}(l, t) - m_2\frac{\partial^2 u}{\partial t^2}(l, t) \tag{E.2}$$

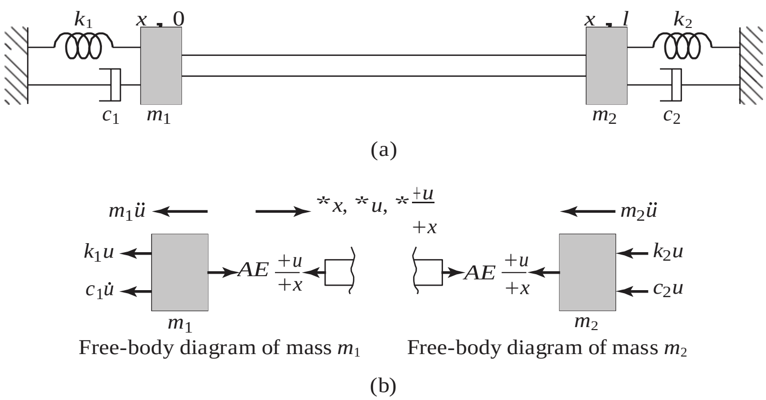


FIGURE 8.8 Bar connected to springs-masses-dampers at ends.

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8.3.2 Orthogonality of Normal Functions

The normal functions for the longitudinal vibration of bars satisfy the orthogonality relation

$$\int_0^l U_i(x)U_j(x)\,dx = 0 \tag{8.53}$$

where $U_i(x)$ and $U_j(x)$ denote the normal functions corresponding to the *i*th and *j*th natural frequencies ω_i and ω_j, respectively. When $u(x,t) = U_i(x)T(t)$ and $u(x,t) = U_j(x)T(t)$ are assumed as solutions, Eq. (8.49) gives

$$c^2\frac{d^2U_i(x)}{dx^2} + \omega_i^2 U_i(x) = 0 \quad \text{or} \quad c^2 U_i''(x) + \omega_i^2 U_i(x) = 0 \tag{8.54}$$

and

$$c^2\frac{d^2U_j(x)}{dx^2} + \omega_j^2 U_j(x) = 0 \quad \text{or} \quad c^2 U_j''(x) + \omega_j^2 U_j(x) = 0 \tag{8.55}$$

where $U_i'' = \frac{d^2U_i}{dx^2}$ and $U_j'' = \frac{d^2U_j}{dx^2}$. Multiplication of Eq. (8.54) by U_j and Eq. (8.55) by U_i gives

$$c^2U_i''U_j + \omega_i^2 U_i U_j = 0 \tag{8.56}$$

$$c^2U_j''U_i + \omega_j^2 U_j U_i = 0 \tag{8.57}$$

Subtraction of Eq. (8.57) from Eq. (8.56) and integration from 0 to *l* results in

$$\int_0^l U_iU_j\,dx = -\frac{c^2}{\omega_i^2 - \omega_j^2}\int_0^l (U_i''U_j - U_j''U_i)\,dx$$

$$= -\frac{c^2}{\omega_i^2 - \omega_j^2}\left[U_i'U_j - U_j'U_i\right]_0^l \tag{8.58}$$

The right-hand side of Eq. (8.58) can be proved to be zero for any combination of boundary conditions. For example, if the bar is fixed at $x = 0$ and free at $x = l$,

$$u(0,t) = 0, \qquad t \geq 0 \qquad \text{or} \qquad U(0) = 0 \tag{8.59}$$

$$\frac{\partial u}{\partial x}(l,t) = 0, \qquad t \geq 0 \qquad \text{or} \qquad U'(l) = 0 \tag{8.60}$$

Thus $(U_i'U_j - U_j'U_i)\big|_{x=l} = 0$ due to U' being zero (Eq. (8.60)) and $(U_i'U_j - U_j'U_i)\big|_{x=0} = 0$ due to U being zero (Eq. (8.59)). Equation (8.58) thus reduces to Eq. (8.53), which is also known as the *orthogonality principle for the normal functions.*

EXAMPLE 8.3 Free Vibrations of a Fixed-Free Bar

Find the natural frequencies and the free-vibration solution of a bar fixed at one end and free at the other.

Solution: Let the bar be fixed at $x = 0$ and free at $x = l$, so that the boundary conditions can be expressed as

$$u(0, t) = 0, \qquad t \geq 0 \tag{E.1}$$

$$\frac{\partial u}{\partial x}(l, t) = 0, \qquad t \geq 0 \tag{E.2}$$

The use of Eq. (E.1) in Eq. (8.51) gives $A = 0$, while the use of Eq. (E.2) gives the frequency equation

$$B\frac{\omega}{c}\cos\frac{\omega l}{c} = 0 \quad \text{or} \quad \cos\frac{\omega l}{c} = 0 \tag{E.3}$$

The eigenvalues or natural frequencies are given by

$$\frac{\omega_n l}{c} = (2n+1)\frac{\pi}{2}, \qquad n = 0, 1, 2, \ldots$$

or

$$\omega_n = \frac{(2n+1)\pi c}{2l}, \qquad n = 0, 1, 2, \ldots \tag{E.4}$$

Thus the total (free-vibration) solution of Eq. (8.49) can be written, using the mode superposition method, as

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2l}\left[C_n \cos\frac{(2n+1)\pi c t}{2l} + D_n \sin\frac{(2n+1)\pi c t}{2l}\right] \end{aligned} \tag{E.5}$$

where the values of the constants C_n and D_n can be determined from the initial conditions, as in Eqs. (8.33) and (8.34):

$$C_n = \frac{2}{l}\int_0^l u_0(x)\sin\frac{(2n+1)\pi x}{2l}\,dx \tag{E.6}$$

$$D_n = \frac{4}{(2n+1)\pi c}\int_0^l \dot{u}_0(x)\sin\frac{(2n+1)\pi x}{2l}\,dx \tag{E.7}$$

*

EXAMPLE 8.4 Natural Frequencies of a Bar Carrying a Mass

Find the natural frequencies of a bar with one end fixed and a mass attached at the other end, as in Fig. 8.9.

Solution: The equation governing the axial vibration of the bar is given by Eq. (8.49) and the solution by Eq. (8.51). The boundary condition at the fixed end $(x = 0)$

$$u(0, t) = 0 \tag{E.1}$$

leads to $A = 0$ in Eq. (8.51). At the end $x = l$, the tensile force in the bar must be equal to the inertia force of the vibrating mass M, and so

$$AE\frac{\partial u}{\partial x}(l, t) = -M\frac{\partial^2 u}{\partial t^2}(l, t) \tag{E.2}$$

With the help of Eq. (8.51), this equation can be expressed as

$$AE\frac{\omega}{c}\cos\frac{\omega l}{c}(C\cos\omega t + D\sin\omega t) = M\omega^2\sin\frac{\omega l}{c}(C\cos\omega t + D\sin\omega t)$$

That is,

$$\frac{AE\omega}{c}\cos\frac{\omega l}{c} = M\omega^2\sin\frac{\omega l}{c}$$

or

$$\alpha\tan\alpha = \beta \tag{E.3}$$

where

$$\alpha = \frac{\omega l}{c} \tag{E.4}$$

and

$$\beta = \frac{AEl}{c^2M} = \frac{A\rho l}{M} = \frac{m}{M} \tag{E.5}$$

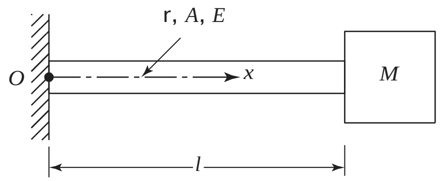


FIGURE 8.9 Bar carrying an end mass.

TABLE 8.1

	Values of the Mass Ratio β				
	0.01	**0.1**	**1.0**	**10.0**	**100.0**
Value of $\alpha_1 = \frac{\omega_1 l}{c}$*	0.1000	0.3113	0.8602	1.4291	1.5549
Value of $\alpha_2 = \frac{\omega_2 l}{c}$*	3.1448	3.1736	3.4267	4.3063	4.6658

where *m* is the mass of the bar. Equation (E.3) is the frequency equation (in the form of a transcendental equation) whose solution gives the natural frequencies of the system. The first two natural frequencies are given in Table 8.1 for different values of the parameter β.

Note: If the mass of the bar is negligible compared to the mass attached, $m \simeq 0$,

$$c = \left(\frac{E}{\rho}\right)^{1/2} = \left(\frac{EAl}{m}\right)^{1/2} \to \infty \quad \text{and} \quad \alpha = \frac{\omega l}{c} \to 0$$

In this case

$$\tan \frac{\omega l}{c} \simeq \frac{\omega l}{c}$$

and the frequency equation (E.3) can be taken as

$$\left(\frac{\omega l}{c}\right)^2 = \beta$$

This gives the approximate value of the fundamental frequency

$$\omega_1 = \frac{c}{l}\beta^{1/2} = \frac{c}{l}\left(\frac{\rho A l}{M}\right)^{1/2} = \left(\frac{EA}{lM}\right)^{1/2} = \left(\frac{g}{\delta_s}\right)^{1/2}$$

where

$$\delta_s = \frac{Mgl}{EA}$$

represents the static elongation of the bar under the action of the load *Mg.*

*

EXAMPLE 8.5 Vibrations of a Bar Subjected to Initial Force

A bar of uniform cross-sectional area *A*, density ρ, modulus of elasticity *E*, and length *l* is fixed at one end and free at the other end. It is subjected to an axial force F_0 at its free end, as shown in Fig. 8.10(a). Study the resulting vibrations if the force F_0 is suddenly removed.

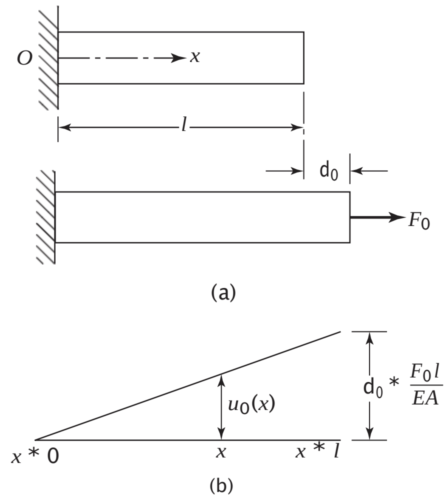


FIGURE 8.10 Bar subjected to an axial force at end.

Solution: The tensile strain induced in the bar due to F_0 is

$$\varepsilon = \frac{F_0}{EA}$$

Thus the displacement of the bar just before the force F_0 is removed (initial displacement) is given by (see Fig. (8.10b))

$$u_0 = u(x, 0) = \varepsilon x = \frac{F_0 x}{EA}, \qquad 0 \le x \le l \tag{E.1}$$

Since the initial velocity is zero, we have

$$\dot{u}_0 = \frac{\partial u}{\partial t}(x, 0) = 0, \qquad 0 \le x \le l \tag{E.2}$$

The general solution of a bar fixed at one end and free at the other end is given by Eq. (E.5) of Example 8.3:

$$\begin{aligned} u(x, t) &= \sum_{n=0}^{\infty} u_n(x, t) \\ &= \sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi x}{2l}\left[C_n \cos\frac{(2n+1)\pi ct}{2l} + D_n \sin\frac{(2n+1)\pi ct}{2l}\right] \end{aligned} \tag{E.3}$$

where C_n and D_n are given by Eqs. (E.6) and (E.7) of Example 8.3. Since $\dot{u}_0 = 0$, we obtain $D_n = 0$. By using the initial displacement of Eq. (E.1) in Eq. (E.6) of Example 8.3, we obtain

$$C_n = \frac{2}{l}\int_0^l \frac{F_0 x}{EA} \sin\frac{(2n+1)\pi x}{2l}\,dx = \frac{8F_0 l}{EA\pi^2}\frac{(-1)^n}{(2n+1)^2} \tag{E.4}$$

Thus the solution becomes

$$u(x,t) = \frac{8F_0 l}{EA\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2l}\cos\frac{(2n+1)\pi ct}{2l} \tag{E.5}$$

Equations (E.3) and (E.5) indicate that the motion of a typical point at $x = x_0$ on the bar is composed of the amplitudes

$$C_n \sin\frac{(2n+1)\pi x_0}{2l}$$

corresponding to the circular frequencies

$$\frac{(2n+1)\pi c}{2l}$$

*

Torsional Vibration of a Shaft or Rod

Figure 8.11 represents a nonuniform shaft subjected to an external torque $f(x, t)$ per unit length. If $\theta(x, t)$ denotes the angle of twist of the cross section, the relation between the torsional deflection and the twisting moment $M_t(x, t)$ is given by [8.8]

$$M_t(x,t) = GJ(x)\frac{\partial\theta}{\partial x}(x,t) \tag{8.61}$$

where G is the shear modulus and $GJ(x)$ is the torsional stiffness, with $J(x)$ denoting the polar moment of inertia of the cross section in the case of a circular section. If the mass polar moment of inertia of the shaft per unit length is I_0, the inertia torque acting on an element of length dx becomes

$$I_0\,dx\,\frac{\partial^2\theta}{\partial t^2}$$

If an external torque $f(x, t)$ acts on the shaft per unit length, the application of Newton's second law yields the equation of motion:

$$(M_t + dM_t) + f\,dx - M_t = I_0\,dx\,\frac{\partial^2\theta}{\partial t^2} \tag{8.62}$$

By expressing dM_t as

$$\frac{\partial M_t}{\partial x}dx$$

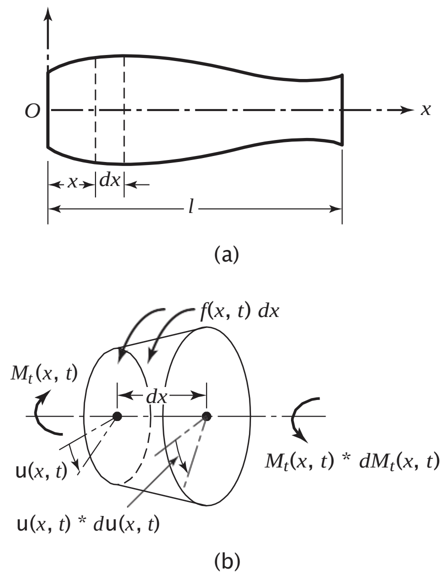


FIGURE 8.11 Torsional vibration of a shaft.

and using Eq. (8.61), the forced torsional vibration equation for a nonuniform shaft can be obtained:

$$\frac{\partial}{\partial x}\left[GJ(x)\frac{\partial\theta}{\partial x}(x, t)\right] + f(x, t) = I_0(x)\frac{\partial^2\theta}{\partial t^2}(x, t) \tag{8.63}$$

For a uniform shaft, Eq. (8.63) takes the form

$$GJ\frac{\partial^2\theta}{\partial x^2}(x, t) + f(x, t) = I_0\frac{\partial^2\theta}{\partial t^2}(x, t) \tag{8.64}$$

which, in the case of free vibration, reduces to

$$c^2\frac{\partial^2\theta}{\partial x^2}(x, t) = \frac{\partial^2\theta}{\partial t^2}(x, t) \tag{8.65}$$

where

$$c = \sqrt{\frac{GJ}{I_0}} \tag{8.66}$$

Notice that Eqs. (8.63) to (8.66) are similar to the equations derived in the cases of transverse vibration of a string and longitudinal vibration of a bar. If the shaft has a uniform cross section, $I_0 = \rho J$. Hence Eq. (8.66) becomes

$$c = \sqrt{\frac{G}{\rho}} \tag{8.67}$$

End Conditions of Shaft	Boundary Conditions	Frequency Equation	Mode Shape (Normal Function)	Natural Frequencies
Fixed-free	$\theta(0, t) = 0$ $\frac{\partial \theta}{\partial x}(l, t) = 0$	$\cos \frac{\omega l}{c} = 0$	$\theta_n(x) = C_n \sin \frac{(2n - 1)\pi x}{2l}$	$\omega_n = \frac{(2n - 1)\pi c}{2l};$ $n = 0, 1, 2, \ldots$
Free-free	$\frac{\partial \theta}{\partial x}(0, t) = 0$ $\frac{\partial \theta}{\partial x}(l, t) = 0$	$\sin \frac{\omega l}{c} = 0$	$\theta_n(x) = C_n \cos \frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 0, 1, 2, \ldots$
Fixed-fixed	$\theta(0, t) = 0$ $\theta(l, t) = 0$	$\sin \frac{\omega l}{c} = 0$	$\theta_n(x) = C_n \cos \frac{n\pi x}{l}$	$\omega_n = \frac{n\pi c}{l};$ $n = 1, 2, 3, \ldots$

FIGURE 8.12 Boundary conditions for uniform shafts (rods) subjected to torsional vibration.

If the shaft is given an angular displacement $\theta_0(x)$ and an angular velocity $\dot{\theta}_0(x)$ at $t = 0$, the initial conditions can be stated as

$$\theta(x, t = 0) = \theta_0(x)$$
$$\frac{\partial \theta}{\partial t}(x, t = 0) = \dot{\theta}_0(x) \tag{8.68}$$

The general solution of Eq. (8.65) can be expressed as

$$\theta(x, t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right) (C \cos \omega t + D \sin \omega t) \tag{8.69}$$

The common boundary conditions for the torsional vibration of uniform shafts are indicated in Fig. 8.12 along with the corresponding frequency equations and the normal functions.

EXAMPLE 8.6

Natural Frequencies of a Milling Cutter

Find the natural frequencies of the plane milling cutter shown in Fig. 8.13 when the free end of the shank is fixed. Assume the torsional rigidity of the shank as GJ and the mass moment of inertia of the cutter as I_0.

Solution: The general solution is given by Eq. (8.69). From this equation, by using the fixed boundary condition $\theta(0, t) = 0$, we obtain $A = 0$. The boundary condition at $x = l$ can be stated as

$$GJ \frac{\partial \theta}{\partial x}(l, t) = -I_0 \frac{\partial^2 \theta}{\partial t^2}(l, t) \tag{E.1}$$

That is,

$$BGJ \frac{\omega}{c} \cos \frac{\omega l}{c} = BI_0 \omega^2 \sin \frac{\omega l}{c}$$

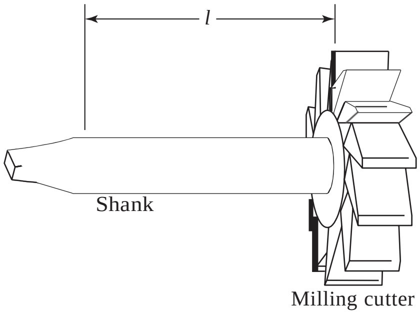


FIGURE 8.13 Plane milling cutter.

or

$$\frac{\omega l}{c}\tan\frac{\omega l}{c} = \frac{J\rho l}{I_0} = \frac{J_{rod}}{I_0} \tag{E.2}$$

where $J_{rod} = J\rho l$. Equation (E.2) can be expressed as

$$\alpha \tan \alpha = \beta \qquad \text{where } \alpha = \frac{\omega l}{c} \qquad \text{and} \qquad \beta = \frac{J_{rod}}{I_0} \tag{E.3}$$

The solution of Eq. (E.3), and thus the natural frequencies of the system, can be obtained as in the case of Example 8.4.

*

8.5 Lateral Vibration of Beams

8.5.1 Equation of Motion

Consider the free-body diagram of an element of a beam shown in Fig. 8.14, where $M(x, t)$ is the bending moment, $V(x, t)$ is the shear force, and $f(x, t)$ is the external force per unit length of the beam. Since the inertia force acting on the element of the beam is

$$\rho A(x)\, dx\, \frac{\partial^2 w}{\partial t^2}(x, t)$$

the force equation of motion in the z direction gives

$$-(V + dV) + f(x, t)\, dx + V = \rho A(x)\, dx\, \frac{\partial^2 w}{\partial t^2}(x, t) \tag{8.70}$$

where ρ is the mass density and $A(x)$ is the cross-sectional area of the beam. The moment equation of motion about the y-axis passing through point O in Fig. 8.14 leads to

$$(M + dM) - (V + dV)\, dx + f(x, t)\, dx\, \frac{dx}{2} - M = 0 \tag{8.71}$$

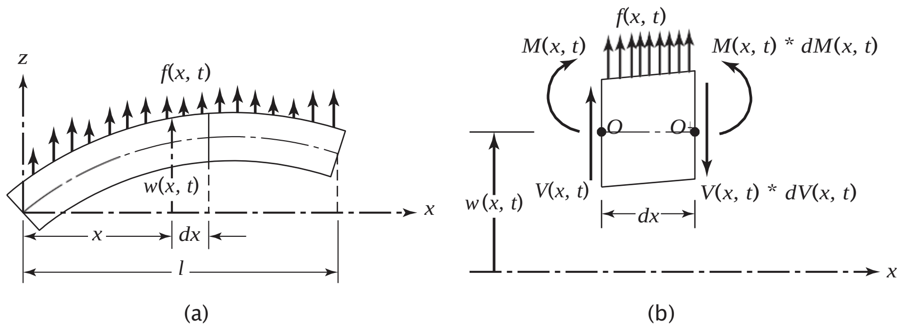


FIGURE 8.14 A beam in bending.

By writing

$$dV = \frac{\partial V}{\partial x}\,dx \quad \text{and} \quad dM = \frac{\partial M}{\partial x}\,dx$$

and disregarding terms involving second powers in dx, Eqs. (8.70) and (8.71) can be written as

$$-\frac{\partial V}{\partial x}(x, t) + f(x, t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x, t) \tag{8.72}$$

$$\frac{\partial M}{\partial x}(x, t) - V(x, t) = 0 \tag{8.73}$$

By using the relation $V = \partial M/\partial x$ from Eq. (8.73), Eq. (8.72) becomes

$$-\frac{\partial^2 M}{\partial x^2}(x, t) + f(x, t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x, t) \tag{8.74}$$

From the elementary theory of bending of beams (also known as the *Euler-Bernoulli* or *thin beam theory*), the relationship between bending moment and deflection can be expressed as [8.8]

$$M(x, t) = EI(x)\frac{\partial^2 w}{\partial x^2}(x, t) \tag{8.75}$$

where E is Young's modulus and $I(x)$ is the moment of inertia of the beam cross section about the y-axis. Inserting Eq. (8.75) into Eq. (8.74), we obtain the equation of motion for the forced lateral vibration of a nonuniform beam:

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}(x, t)\right] + \rho A(x)\frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t) \tag{8.76}$$

For a uniform beam, Eq. (8.76) reduces to

$$EI \frac{\partial^4 w}{\partial x^4}(x, t) + \rho A \frac{\partial^2 w}{\partial t^2}(x, t) = f(x, t) \tag{8.77}$$

For free vibration, $f(x, t) = 0$, and so the equation of motion becomes

$$c^2 \frac{\partial^4 w}{\partial x^4}(x, t) + \frac{\partial^2 w}{\partial t^2}(x, t) = 0 \tag{8.78}$$

where

$$c = \sqrt{\frac{EI}{\rho A}} \tag{8.79}$$

8.5.2 Initial Conditions

Since the equation of motion involves a second-order derivative with respect to time and a fourth-order derivative with respect to x, two initial conditions and four boundary conditions are needed for finding a unique solution for $w(x, t)$. Usually, the values of lateral displacement and velocity are specified as $w_0(x)$ and $\dot{w}_0(x)$ at $t = 0$, so that the initial conditions become

$$\begin{aligned} w(x, t = 0) &= w_0(x) \\ \frac{\partial w}{\partial t}(x, t = 0) &= \dot{w}_0(x) \end{aligned} \tag{8.80}$$

8.5.3 Free Vibration

The free-vibration solution can be found using the method of separation of variables as

$$w(x, t) = W(x)T(t) \tag{8.81}$$

Substituting Eq. (8.81) into Eq. (8.78) and rearranging leads to

$$\frac{c^2}{W(x)} \frac{d^4 W(x)}{dx^4} = -\frac{1}{T(t)} \frac{d^2 T(t)}{dt^2} = a = \omega^2 \tag{8.82}$$

where $a = \omega^2$ is a positive constant (see Problem 8.45). Equation (8.82) can be written as two equations:

$$\frac{d^4 W(x)}{dx^4} - \beta^4 W(x) = 0 \tag{8.83}$$

$$\frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0 \tag{8.84}$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \tag{8.85}$$

The solution of Eq. (8.84) can be expressed as

$$T(t) = A\cos \omega t + B\sin \omega t \tag{8.86}$$

where A and B are constants that can be found from the initial conditions. For the solution of Eq. (8.83), we assume

$$W(x) = Ce^{sx} \tag{8.87}$$

where C and s are constants, and derive the auxiliary equation as

$$s^4 - \beta^4 = 0 \tag{8.88}$$

The roots of this equation are

$$s_{1,2} = \pm\beta, \qquad s_{3,4} = \pm i\beta \tag{8.89}$$

Hence the solution of Eq. (8.83) becomes

$$W(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x} \tag{8.90}$$

where C_1, C_2, C_3, and C_4 are constants. Equation (8.90) can also be expressed as

$$W(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x \tag{8.91}$$

or

$$\begin{aligned} W(x) = {} & C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) \\ & + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x) \end{aligned} \tag{8.92}$$

where C_1, C_2, C_3, and C_4, in each case, are different constants. The constants C_1, C_2, C_3, and C_4 can be found from the boundary conditions. The natural frequencies of the beam are computed from Eq. (8.85) as

$$\omega = \beta^2\sqrt{\frac{EI}{\rho A}} = (\beta l)^2\sqrt{\frac{EI}{\rho A l^4}} \tag{8.93}$$

The function $W(x)$ is known as the *normal mode* or *characteristic function* of the beam and ω is called the *natural frequency of vibration.* For any beam, there will be an infinite number of normal modes with one natural frequency associated with each normal mode. The unknown constants C_1 to C_4 in Eq. (8.91) or (8.92) and the value of β in Eq. (8.93) can be determined from the boundary conditions of the beam as indicated below.

8.5.4 Boundary Conditions

The common boundary conditions are as follows:

1. *Free end:*

$$\text{Bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0$$

$$\text{Shear force} = \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0 \tag{8.94}$$

2. *Simply supported (pinned) end:*

$$\text{Deflection} = w = 0, \qquad \text{Bending moment} = EI\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.95}$$

3. *Fixed (clamped) end:*

$$\text{Deflection} = 0, \qquad \text{Slope} = \frac{\partial w}{\partial x} = 0 \tag{8.96}$$

The frequency equations, the mode shapes (normal functions), and the natural frequencies for beams with common boundary conditions are given in Fig. 8.15 [8.13, 8.17]. We shall now consider some other possible boundary conditions for a beam.

4. *End connected to a linear spring, damper, and mass* (Fig. 8.16(a)): When the end of a beam undergoes a transverse displacement w and slope $\partial w/\partial x$. with velocity $\partial w/\partial t$ and acceleration $\partial^2 w/\partial t^2$, the resisting forces due to the spring, damper, and mass are proportional to w, $\partial w/\partial t$, and $\partial^2 w/\partial t^2$, respectively. This resisting force is balanced by the shear force at the end. Thus

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = a\left[kw + c\frac{\partial w}{\partial t} + m\frac{\partial^2 w}{\partial t^2}\right] \tag{8.97}$$

where $a = -1$ for the left end and $+1$ for the right end of the beam. In addition, the bending moment must be zero; hence

$$EI\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.98}$$

5. *End connected to a torsional spring, torsional damper, and rotational inertia* (Fig. 8.16(b)): In this case, the boundary conditions are

$$EI\frac{\partial^2 w}{\partial x^2} = a\left[k_t\frac{\partial w}{\partial x} + c_t\frac{\partial^2 w}{\partial x \partial t} + I_0\frac{\partial^3 w}{\partial x \partial t^2}\right] \tag{8.99}$$

where $a = +1$ for the left end and -1 for the right end of the beam, and

$$\frac{\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = 0 \tag{8.100}$$

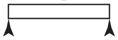
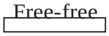
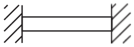
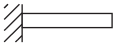
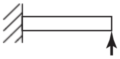
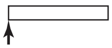
End Conditions of Beam	Frequency Equation	Mode Shape (Normal Function)	Value of $\beta_n l$
Pinned-pinned 	$\sin\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x]$	$\beta_1 l = \pi$ $\beta_2 l = 2\pi$ $\beta_3 l = 3\pi$ $\beta_4 l = 4\pi$
Free-free 	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sin\beta_n x + \sinh\beta_n x + \alpha_n(\cos\beta_n x + \cosh\beta_n x)]$ where $\alpha_n = \frac{\sin\beta_n l - \sinh\beta_n l}{\cosh\beta_n l - \cos\beta_n l}$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$ ($\beta l = 0$ for rigid-body mode)
Fixed-fixed 	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sinh\beta_n x - \sin\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \frac{\sinh\beta_n l - \sin\beta_n l}{\cos\beta_n l - \cosh\beta_n l}$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Fixed-free 	$\cos\beta_n l \cdot \cosh\beta_n l = -1$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x - \alpha_n(\cos\beta_n x - \cosh\beta_n x)]$ where $\alpha_n = \frac{\sin\beta_n l + \sinh\beta_n l}{\cos\beta_n l + \cosh\beta_n l}$	$\beta_1 l = 1.875104$ $\beta_2 l = 4.694091$ $\beta_3 l = 7.854757$ $\beta_4 l = 10.995541$
Fixed-pinned 	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l - \sinh\beta_n l}{\cos\beta_n l - \cosh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$
Pinned-free 	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x + \alpha_n \sinh\beta_n x]$ where $\alpha_n = \left(\frac{\sin\beta_n l}{\sinh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$ ($\beta l = 0$ for rigid-body mode)

FIGURE 8.15 Common boundary conditions for the transverse vibration of a beam.

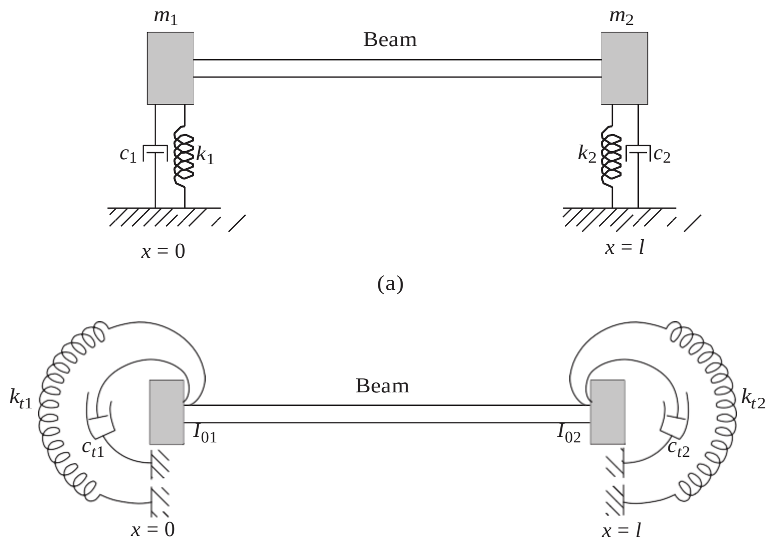
8.5.5 Orthogonality of Normal Functions

The normal functions $W(x)$ satisfy Eq. (8.83):

$$c^2 \frac{d^4W}{dx^4}(x) - \omega^2 W(x) = 0 \tag{8.101}$$

Let $W_i(x)$ and $W_j(x)$ be the normal functions corresponding to the natural frequencies ω_i and $\omega_j (i \neq j)$, so that

$$c^2 \frac{d^4W_i}{dx^4} - \omega_i^2 W_i = 0 \tag{8.102}$$



(b)

FIGURE 8.16 Beams connected with springs-dampers-masses at ends.

and

$$c^2 \frac{d^4 W_j}{dx^4} - \omega_j^2 W_j = 0 \tag{8.103}$$

Multiplying Eq. (8.102) by W_j and Eq. (8.103) by W_i, subtracting the resulting equations one from the other, and integrating from 0 to l gives

$$\int_0^l \left(c^2 \frac{d^4 W_i}{dx^4} W_j - \omega_i^2 W_i W_j \right) dx - \int_0^l \left(c^2 \frac{d^4 W_j}{dx^4} W_i - \omega_j^2 W_j W_i \right) dx = 0$$

or

$$\int_0^l W_i W_j \, dx = -\frac{c^2}{\omega_i^2 - \omega_j^2} \int_0^l (W_i'''' W_j - W_i W_j'''') \, dx \tag{8.104}$$

where a prime indicates differentiation with respect to x. The right-hand side of Eq. (8.104) can be evaluated using integration by parts to obtain

$$\int_0^l W_i W_j \, dx = -\frac{c^2}{\omega_i^2 - \omega_j^2} \left[W_i W_j''' - W_j W_i''' + W_j' W_i'' - W_i' W_j'' \right]_0^l \tag{8.105}$$

The right-hand side of Eq. (8.105) can be shown to be zero for any combination of free, fixed, or simply supported end conditions. At a free end, the bending moment and shear force are equal to zero so that

$$W'' = 0, \qquad W''' = 0 \tag{8.106}$$

For a fixed end, the deflection and slope are zero:

$$W = 0, \qquad W' = 0 \tag{8.107}$$

At a simply supported end, the bending moment and deflection are zero:

$$W'' = 0, \qquad W = 0 \tag{8.108}$$

Since each term on the right-hand side of Eq. (8.105) is zero at $x = 0$ or $x = l$ for any combination of the boundary conditions in Eqs. (8.106) to (8.108), Eq. (8.105) reduces to

$$\int_0^l W_i W_j \, dx = 0 \tag{8.109}$$

which proves the orthogonality of normal functions for the transverse vibration of beams.

EXAMPLE 8.7 Natural Frequencies of a Fixed-Pinned Beam

Determine the natural frequencies of vibration of a uniform beam fixed at $x = 0$ and simply supported at $x = l$.

Solution: The boundary conditions can be stated as

$$W(0) = 0 \tag{E.1}$$

$$\frac{dW}{dx}(0) = 0 \tag{E.2}$$

$$W(l) = 0 \tag{E.3}$$

$$EI\frac{d^2W}{dx^2}(l) = 0 \quad \text{or} \quad \frac{d^2W}{dx^2}(l) = 0 \tag{E.4}$$

Condition (E.1) leads to

$$C_1 + C_3 = 0 \tag{E.5}$$

in Eq. (8.91), while Eqs. (E.2) and (8.91) give

$$\left.\frac{dW}{dx}\right|_{x=0} = \beta[-C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x]_{x=0} = 0$$

or

$$\beta[C_2 + C_4] = 0 \tag{E.6}$$

Thus the solution, Eq. (8.91), becomes

$$W(x) = C_1(\cos \beta x - \cosh \beta x) + C_2(\sin \beta x - \sinh \beta x) \tag{E.7}$$

Applying conditions (E.3) and (E.4) to Eq. (E.7) yields

$$C_1(\cos \beta l - \cosh \beta l) + C_2(\sin \beta l - \sinh \beta l) = 0 \tag{E.8}$$

$$-C_1(\cos \beta l + \cosh \beta l) - C_2(\sin \beta l + \sinh \beta l) = 0 \tag{E.9}$$

For a nontrivial solution of C_1 and C_2, the determinant of their coefficients must be zero—that is,

$$\begin{vmatrix} (\cos \beta l - \cosh \beta l) & (\sin \beta l - \sinh \beta l) \\ -(\cos \beta l + \cosh \beta l) & -(\sin \beta l + \sinh \beta l) \end{vmatrix} = 0 \tag{E.10}$$

Expanding the determinant gives the frequency equation

$$\cos \beta l \sinh \beta l - \sin \beta l \cosh \beta l = 0$$

or

$$\tan \beta l = \tanh \beta l \tag{E.11}$$

The roots of this equation, $\beta_n l$, give the natural frequencies of vibration

$$\omega_n = (\beta_n l)^2 \left(\frac{EI}{\rho A l^4}\right)^{1/2}, \qquad n = 1, 2, \ldots \tag{E.12}$$

where the values of $\beta_n l$, $n = 1, 2, \ldots$ satisfying Eq. (E.11) are given in Fig. 8.15. If the value of C_2 corresponding to β_n is denoted as C_{2n}, it can be expressed in terms of C_{1n} from Eq. (E.8) as

$$C_{2n} = -C_{1n}\left(\frac{\cos \beta_n l - \cosh \beta_n l}{\sin \beta_n l - \sinh \beta_n l}\right) \tag{E.13}$$

Hence Eq. (E.7) can be written as

$$W_n(x) = C_{1n}\left[(\cos \beta_n x - \cosh \beta_n x) - \left(\frac{\cos \beta_n l - \cosh \beta_n l}{\sin \beta_n l - \sinh \beta_n l}\right)(\sin \beta_n x - \sinh \beta_n x)\right] \tag{E.14}$$

The normal modes of vibration can be obtained by the use of Eq. (8.81)

$$w_n(x, t) = W_n(x)(A_n \cos \omega_n t + B_n \sin \omega_n t) \tag{E.15}$$

with $W_n(x)$ given by Eq. (E.14). The general or total solution of the fixed-simply supported beam can be expressed by the sum of the normal modes:

$$w(x, t) = \sum_{n=1}^{\infty} w_n(x, t) \tag{E.16}$$

*

8.5.6 Forced Vibration

The forced-vibration solution of a beam can be determined using the mode superposition principle. For this, the deflection of the beam is assumed as

$$w(x, t) = \sum_{n=1}^{\infty} W_n(x) q_n(t) \tag{8.110}$$

where $W_n(x)$ is the nth normal mode or characteristic function satisfying the differential equation (Eq. 8.101)

$$EI \frac{d^4 W_n(x)}{dx^4} - \omega_n^2 \rho A W_n(x) = 0; \qquad n = 1, 2, \ldots \tag{8.111}$$

and $q_n(t)$ is the generalized coordinate in the nth mode. By substituting Eq. (8.110) into the forced-vibration equation, Eq. (8.77), we obtain

$$EI \sum_{n=1}^{\infty} \frac{d^4 W_n(x)}{dx^4} q_n(t) + \rho A \sum_{n=1}^{\infty} W_n(x) \frac{d^2 q_n(t)}{dt^2} = f(x, t) \tag{8.112}$$

In view of Eq. (8.111), Eq. (8.112) can be written as

$$\sum_{n=1}^{\infty} \omega_n^2 W_n(x) q_n(t) + \sum_{n=1}^{\infty} W_n(x) \frac{d^2 q_n(t)}{dt^2} = \frac{1}{\rho A} f(x, t) \tag{8.113}$$

By multiplying Eq. (8.113) throughout by $W_m(x)$, integrating from 0 to l, and using the orthogonality condition, Eq. (8.109), we obtain

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 q_n(t) = \frac{1}{\rho A b} Q_n(t) \tag{8.114}$$

where $Q_n(t)$ is called the generalized force corresponding to $q_n(t)$

$$Q_n(t) = \int_0^l f(x, t) W_n(x)\, dx \tag{8.115}$$

and the constant b is given by

$$b = \int_0^l W_n^2(x)\, dx \tag{8.116}$$

Equation (8.114) can be identified to be, essentially, the same as the equation of motion of an undamped single-degree-of-freedom system. Using the Duhamel integral, the solution of Eq. (8.114) can be expressed as

$$q_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau)\, d\tau \tag{8.117}$$

where the first two terms on the right-hand side of Eq. (8.117) represent the transient or free vibration (resulting from the initial conditions) and the third term denotes the steady-state vibration (resulting from the forcing function). Once Eq. (8.117) is solved for $n = 1, 2, \ldots$, the total solution can be determined from Eq. (8.110).

EXAMPLE 8.8

Forced Vibration of a Simply Supported Beam

Find the steady-state response of a pinned-pinned beam subject to a harmonic force $f(x, t) = f_0 \sin \omega t$ applied at $x = a$, as shown in Fig. 8.17.

Solution: *Approach:* Mode superposition method.

The normal mode functions of a pinned-pinned beam are given by (see Fig. 8.15; also Problem 8.33)

$$W_n(x) = \sin \beta_n x = \sin \frac{n\pi x}{l} \tag{E.1}$$

where

$$\beta_n l = n\pi \tag{E.2}$$

The generalized force $Q_n(t)$, given by Eq. (8.115), becomes

$$Q_n(t) = \int_0^l f(x, t) \sin \beta_n x \, dx = f_0 \sin \frac{n\pi a}{l} \sin \omega t \tag{E.3}$$

The steady-state response of the beam is given by Eq. (8.117)

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau \tag{E.4}$$

where

$$b = \int_0^l W_n^2(x) \, dx = \int_0^l \sin^2 \beta_n x \, dx = \frac{l}{2} \tag{E.5}$$

The solution of Eq. (E.4) can be expressed as

$$q_n(t) = \frac{2 f_0}{\rho A l} \frac{\sin \frac{n\pi a}{l}}{\omega_n^2 - \omega^2} \sin \omega t \tag{E.6}$$

Thus the response of the beam is given by Eq. (8.110):

$$w(x, t) = \frac{2 f_0}{\rho A l} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 - \omega^2} \sin \frac{n\pi a}{l} \sin \frac{n\pi x}{l} \sin \omega t \tag{E.7}$$

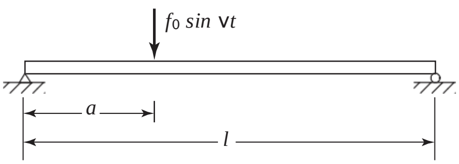


FIGURE 8.17 Pinned-pinned beam under harmonic force.

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8.5.7 Effect of Axial Force

The problem of vibrations of a beam under the action of axial force finds application in the study of vibrations of cables and guy wires. For example, although the vibrations of a cable can be found by treating it as an equivalent string, many cables have failed due to fatigue caused by alternating flexure. The alternating flexure is produced by the regular shedding of vortices from the cable in a light wind. We must therefore consider the effects of axial force and bending stiffness on lateral vibrations in the study of fatigue failure of cables.

To find the effect of an axial force $P(x, t)$ on the bending vibrations of a beam, consider the equation of motion of an element of the beam, as shown in Fig. 8.18. For the vertical motion, we have

$$-(V + dV) + f\,dx + V + (P + dP)\sin(\theta + d\theta) - P\sin\theta = \rho A\,dx\,\frac{\partial^2 w}{\partial t^2} \tag{8.118}$$

and for the rotational motion about 0,

$$(M + dM) - (V + dV)\,dx + f\,dx\,\frac{dx}{2} - M = 0 \tag{8.119}$$

For small deflections,

$$\sin(\theta + d\theta) \approx \theta + d\theta = \theta + \frac{\partial \theta}{\partial x}dx = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx$$

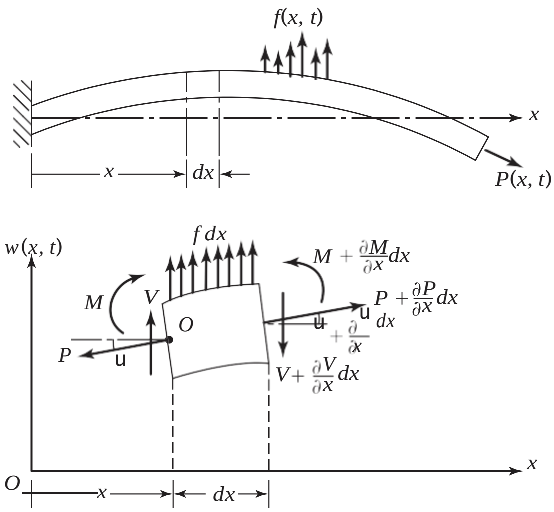

FIGURE 8.18 An element of a beam under axial load.

With this, Eqs. (8.118), (8.119), and (8.75) can be combined to obtain a single differential equation of motion:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 w}{\partial x^2}\right] + \rho A\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = f \tag{8.120}$$

For the free vibration of a uniform beam, Eq. (8.120) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - P\frac{\partial^2 w}{\partial x^2} = 0 \tag{8.121}$$

The solution of Eq. (8.121) can be obtained using the method of separation of variables as

$$w(x, t) = W(x)(A\cos\omega t + B\sin\omega t) \tag{8.122}$$

Substitution of Eq. (8.122) into Eq. (8.121) gives

$$EI\frac{d^4W}{dx^4} - P\frac{d^2W}{dx^2} - \rho A\omega^2 W = 0 \tag{8.123}$$

By assuming the solution $W(x)$ to be

$$W(x) = Ce^{sx} \tag{8.124}$$

in Eq. (8.123), the auxiliary equation can be obtained:

$$s^4 - \frac{P}{EI}s^2 - \frac{\rho A\omega^2}{EI} = 0 \tag{8.125}$$

The roots of Eq. (8.125) are

$$s_1^2, s_2^2 = \frac{P}{2EI} \pm \left(\frac{P^2}{4E^2I^2} + \frac{\rho A\omega^2}{EI}\right)^{1/2} \tag{8.126}$$

and so the solution can be expressed as (with absolute value of s_2)

$$W(x) = C_1\cosh s_1x + C_2\sinh s_1x + C_3\cos s_2x + C_4\sin s_2x \tag{8.127}$$

where the constants C_1 to C_4 are to be determined from the boundary conditions.

EXAMPLE 8.9 Beam Subjected to an Axial Compressive Force

Find the natural frequencies of a simply supported beam subjected to an axial compressive force.

Solution: The boundary conditions are

$$W(0) = 0 \tag{E.1}$$

$$\frac{d^2W}{dx^2}(0) = 0 \tag{E.2}$$

$$W(l) = 0 \tag{E.3}$$

$$\frac{d^2W}{dx^2}(l) = 0 \tag{E.4}$$

Equations (E.1) and (E.2) require that $C_1 = C_3 = 0$ in Eq. (8.127), and so

$$W(x) = C_2 \sinh s_1 x + C_4 \sin s_2 x \tag{E.5}$$

The application of Eqs. (E.3) and (E.4) to Eq. (E.5) leads to

$$\sinh s_1 l \cdot \sin s_2 l = 0 \tag{E.6}$$

Since $\sinh s_1 l > 0$ for all values of $s_1 l \neq 0$, the only roots to this equation are

$$s_2 l = n\pi, \qquad n = 0, 1, 2, \ldots \tag{E.7}$$

Thus Eqs. (E.7) and (8.126) give the natural frequencies of vibration:

$$\omega_n = \frac{\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}}\left\{n^4 + \frac{n^2 P l^2}{\pi^2 EI}\right\}^{1/2} \tag{E.8}$$

Since the axial force P is compressive, P is negative. Further, from strength of materials, the smallest Euler buckling load for a simply supported beam is given by [8.9]

$$P_{\text{cri}} = \frac{\pi^2 EI}{l^2} \tag{E.9}$$

Thus Eq. (E.8) can be written as

$$\omega_n = \frac{\pi^2}{l^2}\left(\frac{EI}{\rho A}\right)^{1/2}\left\{n^4 - n^2 \frac{P}{P_{\text{cri}}}\right\}^{1/2} \tag{E.10}$$

The following observations can be made from the present example:

1. If $P = 0$, the natural frequency will be same as that of a simply supported beam given in Fig. 8.15.
2. If $EI = 0$, the natural frequency (see Eq. (E.8)) reduces to that of a taut string.
3. If $P > 0$, the natural frequency increases as the tensile force stiffens the beam.
4. As $P \to P_{\text{cri}}$, the natural frequency approaches zero for $n = 1$.

■

8.5.8 Effects of Rotary Inertia and Shear Deformation

If the cross-sectional dimensions are not small compared to the length of the beam, we need to consider the effects of rotary inertia and shear deformation. The procedure, presented by Timoshenko [8.10], is known as the *thick beam theory* or *Timoshenko beam theory*. Consider the element of the beam shown in Fig. 8.19. If the effect of shear deformation is disregarded, the tangent to the deflected center line $O'T$ coincides with the normal to the face $Q'R'$ (since cross sections normal to the center line remain normal even

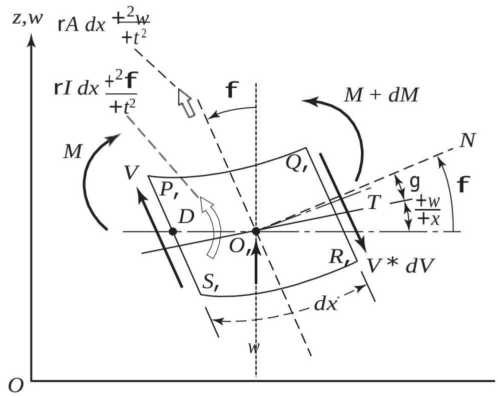

FIGURE 8.19 An element of Timoshenko beam.

after deformation). Due to shear deformation, the tangent to the deformed center line $O'T$ will not be perpendicular to the face $Q'R'$. The angle γ between the tangent to the deformed center line $(O'T)$ and the normal to the face $(O'N)$ denotes the shear deformation of the element. Since positive shear on the right face $Q'R'$ acts downward, we have, from Fig. 8.19,

$$\gamma = \phi - \frac{\partial w}{\partial x} \tag{8.128}$$

where ϕ denotes the slope of the deflection curve due to bending deformation alone. Note that because of shear alone, the element undergoes distortion but no rotation.

The bending moment M and the shear force V are related to ϕ and w by the formulas[2]

$$M = EI\,\frac{\partial \phi}{\partial x} \tag{8.129}$$

and

$$V = kAG\gamma = kAG\left(\phi - \frac{\partial w}{\partial x}\right) \tag{8.130}$$

where G denotes the modulus of rigidity of the material of the beam and k is a constant, also known as *Timoshenko's shear coefficient*, which depends on the shape of

[2]Equation (8.129) is similar to Eq. (8.75). Equation (8.130) can be obtained as follows:

$$\text{Shear force} = \text{Shear stress} \times \text{Area} = \text{Shear strain} \times \text{Shear modulus} \times \text{Area}$$

or

$$V = \gamma G A$$

This equation is modified as $V = kAG\gamma$ by introducing a factor k on the right-hand side to take care of the shape of the cross section.

the cross section. For a rectangular section the value of k is 5/6; for a circular section it is 9/10 [8.11].

The equations of motion for the element shown in Fig. 8.19 can be derived as follows:

1. For translation in the z direction:

$$-[V(x,t) + dV(x,t)] + f(x,t)\,dx + V(x,t) = \rho A(x)\,dx\,\frac{\partial^2 w}{\partial t^2}(x,t) \quad \leftarrow \text{Translational inertia of the element} \tag{8.131}$$

2. For rotation about a line passing through point D and parallel to the y-axis:

$$[M(x,t) + dM(x,t)] + [V(x,t) + dV(x,t)]\,dx + f(x,t)\,dx\,\frac{dx}{2} - M(x,t) = \rho I(x)\,dx\,\frac{\partial^2 \phi}{\partial t^2} \quad \leftarrow \text{Rotary inertia of the element} \tag{8.132}$$

Using the relations

$$dV = \frac{\partial V}{\partial x}dx \qquad \text{and} \qquad dM = \frac{\partial M}{\partial x}dx$$

along with Eqs. (8.129) and (8.130) and disregarding terms involving second powers in dx, Eqs. (8.131) and (8.132) can be expressed as

$$-kAG\left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + f(x,t) = \rho A \frac{\partial^2 w}{\partial t^2} \tag{8.133}$$

$$EI\frac{\partial^2 \phi}{\partial x^2} - kAG\left(\phi - \frac{\partial w}{\partial x}\right) = \rho I \frac{\partial^2 \phi}{\partial t^2} \tag{8.134}$$

By solving Eq. (8.133) for $\partial \phi/\partial x$ and substituting the result in Eq. (8.134), we obtain the desired equation of motion for the forced vibration of a uniform beam:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\,\partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4}$$

$$+\frac{EI}{kAG}\frac{\partial^2 f}{\partial x^2} - \frac{\rho I}{kAG}\frac{\partial^2 f}{\partial t^2} - f = 0 \tag{8.135}$$

For free vibration, $f = 0$, and Eq. (8.135) reduces to

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{8.136}$$

The following boundary conditions are to be applied in the solution of Eq. (8.135) or (8.136):

1. Fixed end:

$$\phi = w = 0$$

2. Simply supported end:

$$EI\frac{\partial \phi}{\partial x} = w = 0$$

3. Free end:

$$kAG\left(\frac{\partial w}{\partial x} - \phi\right) = EI\frac{\partial \phi}{\partial x} = 0$$

EXAMPLE 8.10

Natural Frequencies of a Simply Supported Beam

Determine the effects of rotary inertia and shear deformation on the natural frequencies of a simply supported uniform beam.

Solution: By defining

$$\alpha^2 = \frac{EI}{\rho A} \quad \text{and} \quad r^2 = \frac{I}{A} \tag{E.1}$$

Eq. (8.136) can be written as

$$\alpha^2\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - r^2\left(1 + \frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho r^2}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{E.2}$$

We can express the solution of Eq. (E.2) as

$$w(x, t) = C\sin\frac{n\pi x}{l}\cos\omega_n t \tag{E.3}$$

which satisfies the necessary boundary conditions at $x = 0$ and $x = l$. Here, C is a constant and ω_n is the nth natural frequency. By substituting Eq. (E.3) into Eq. (E.2), we obtain the frequency equation:

$$\omega_n^4\left(\frac{\rho r^2}{kG}\right) - \omega_n^2\left(1 + \frac{n^2\pi^2r^2}{l^2} + \frac{n^2\pi^2r^2}{l^2}\frac{E}{kG}\right) + \left(\frac{\alpha^2n^4\pi^4}{l^4}\right) = 0 \tag{E.4}$$

It can be seen that Eq. (E.4) is a quadratic equation in ω_n^2, and for any given n there are two values of ω_n that satisfy Eq. (E.4). The smaller value corresponds to the bending deformation mode, while the larger one corresponds to the shear deformation mode.

The values of the ratio of ω_n given by Eq. (E.4) to the natural frequency given by the classical theory (in Fig. 8.15) are plotted for three values of E/kG in Fig. 8.20 [8.22].[3]

Note the following aspects of rotary inertia and shear deformation:

1. If the effect of rotary inertia alone is considered, the resulting equation of motion does not contain any term involving the shear coefficient k. Hence we obtain (from Eq. (8.136)):

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w}{\partial x^2\,\partial t^2} = 0 \tag{E.5}$$

   In this case the frequency equation (E.4) reduces to

$$\omega_n^2 = \frac{\alpha^2n^4\pi^4}{l^4\left(1 + \dfrac{n^2\pi^2r^2}{l^2}\right)} \tag{E.6}$$

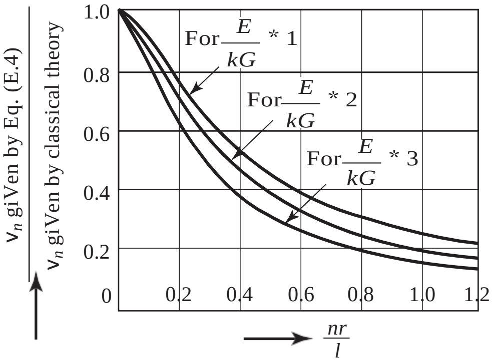

FIGURE 8.20 Variation of frequency.

[3] The theory used for the derivation of the equation of motion (8.76), which disregards the effects of rotary inertia and shear deformation, is called the *classical* or *Euler-Bernoulli* or *thin beam theory*.

2. If the effect of shear deformation alone is considered, the resulting equation of motion does not contain the terms originating from $\rho I(\partial^2\phi/\partial t^2)$ in Eq. (8.134). Thus we obtain the equation of motion

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \frac{EI\rho}{kG}\frac{\partial^4 w}{\partial x^2\,\partial t^2} = 0 \tag{E.7}$$

and the corresponding frequency equation

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4\left(1 + \dfrac{n^2\pi^2 r^2}{l^2}\dfrac{E}{kG}\right)} \tag{E.8}$$

3. If both the effects of rotary inertia and shear deformation are disregarded, Eq. (8.136) reduces to the classical equation of motion, Eq. (8.78),

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \tag{E.9}$$

and Eq. (E.4) to

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4} \tag{E.10}$$

*

8.5.9 Other Effects

The transverse vibration of tapered beams is presented in references [8.12, 8.14]. The natural frequencies of continuous beams are discussed by Wang [8.15]. The dynamic response of beams resting on elastic foundation is considered in reference [8.16]. The effect of support flexibility on the natural frequencies of beams is presented in [8.18, 8.19]. A treatment of the problem of natural vibrations of a system of elastically connected Timoshenko beams is given in reference [8.20]. A comparison of the exact and approximate solutions of vibrating beams is made by Hutchinson [8.30]. The steady-state vibration of damped beams is considered in reference [8.21].

8.6 Vibration of Membranes

A membrane is a plate that is subjected to tension and has negligible bending resistance. Thus a membrane bears the same relationship to a plate as a string bears to a beam. A drumhead is an example of a membrane.

8.6.1 Equation of Motion

To derive the equation of motion of a membrane, consider the membrane to be bounded by a plane curve S in the xy-plane, as shown in Fig. 8.21. Let $f(x, y, t)$ denote the pressure loading acting in the z direction and P the intensity of tension at a point that is equal to the product of the tensile stress and the thickness of the membrane. The magnitude of P is usually

This shows that the addition of a 10.0561-oz weight in the left plane at 145.5548° and a 5.8774-oz weight in the right plane at 248.2559° from the reference position will balance the turbine rotor. It is implied that the balance weights are added at the same radial distance as the trial weights. If a balance weight is to be located at a different radial position, the required balance weight is to be modified in inverse proportion to the radial distance from the axis of rotation.

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9.5 Whirling of Rotating Shafts

In the previous section, the rotor system—the shaft as well as the rotating body—was assumed to be rigid. However, in many practical applications, such as turbines, compressors, electric motors, and pumps, a heavy rotor is mounted on a lightweight, flexible shaft that is supported in bearings. There will be unbalance in all rotors due to manufacturing errors. These unbalances as well as other effects, such as the stiffness and damping of the shaft, gyroscopic effects, and fluid friction in bearings, will cause a shaft to bend in a complicated manner at certain rotational speeds, known as the whirling, whipping, or critical speeds. Whirling is defined as the rotation of the plane made by the line of centers of the bearings and the bent shaft. We consider the aspects of modeling the rotor system, critical speeds, response of the system, and stability in this section [9.13–9.14].

9.5.1 Equations of Motion

Consider a shaft supported by two bearings and carrying a rotor or disc of mass m at the middle, as shown in Fig. 9.11. We shall assume that the rotor is subjected to a steady-state excitation due to mass unbalance. The forces acting on the rotor are the inertia force due to the acceleration of the mass center, the spring force due to the elasticity of the shaft, and the external and internal damping forces.[3]

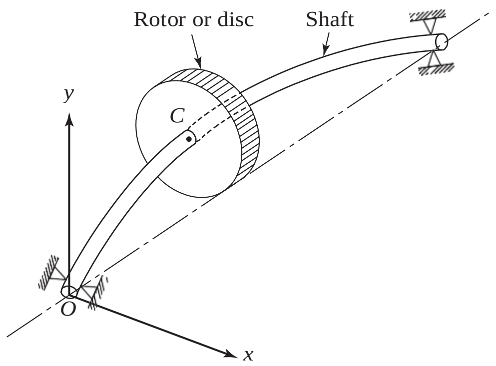

FIGURE 9.11 Shaft carrying a rotor.

[3] Any rotating system responds in two different ways to damping or friction forces, depending upon whether the forces rotate with the shaft or not. When the positions at which the forces act remain fixed in space, as in the case of damping forces (which cause energy losses) in the bearing support structure, the damping is called *stationary* or *external damping*. On the other hand, if the positions at which they act rotate with the shaft in space, as in the case of internal friction of the shaft material, the damping is called *rotary* or *internal damping*.

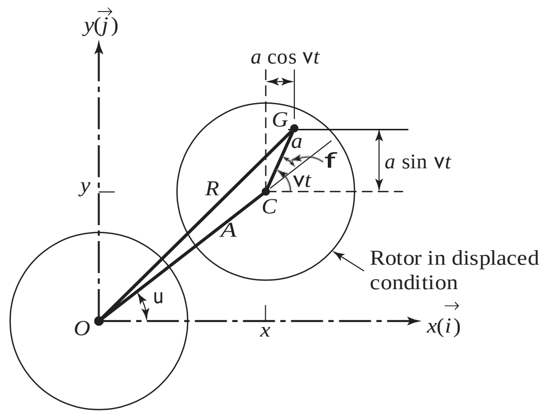

FIGURE 9.12 Rotor with eccentricity.

Let O denote the equilibrium position of the shaft when balanced perfectly, as shown in Fig. 9.12. The shaft (line CG) is assumed to rotate with a constant angular velocity ω. During rotation, the rotor deflects radially by a distance $A = OC$ (in steady state). The rotor (disc) is assumed to have an eccentricity a so that its mass center (center of gravity) G is at a distance a from the geometric center, C. We use a fixed coordinate system (x and y fixed to the earth) with O as the origin for describing the motion of the system. The angular velocity of the line OC, $\dot{\theta} = d\theta/dt$, is known as the whirling speed and, in general, is not equal to ω. The equations of motion of the rotor (mass m) can be written as

$$\begin{aligned}\text{Inertia force } (\vec{F}_i) = {} & \text{Elastic force } (\vec{F}_e) \\ & + \text{Internal damping force } (\vec{F}_{di}) \\ & + \text{External damping force } (\vec{F}_{de})\end{aligned} \tag{9.25}$$

The various forces in Eq. (9.25) can be expressed as follows:

$$\text{Inertia force: } \vec{F}_i = m\ddot{\vec{R}} \tag{9.26}$$

where $\vec{R}$ denotes the radius vector of the mass center G given by

$$\vec{R} = (x + a\cos\omega t)\vec{i} + (y + a\sin\omega t)\vec{j} \tag{9.27}$$

with x and y representing the coordinates of the geometric center C and $\vec{i}$ and $\vec{j}$ denoting the unit vectors along the x and y coordinates, respectively. Equations (9.26) and (9.27) lead to

$$\vec{F}_i = m[(\ddot{x} - a\omega^2\cos\omega t)\vec{i} + (\ddot{y} - a\omega^2\sin\omega t)\vec{j}] \tag{9.28}$$

$$\text{Elastic force: } \vec{F}_e = -k(x\vec{i} + y\vec{j}) \tag{9.29}$$

where k is the stiffness of the shaft.

$$\text{Internal damping force: } \vec{F}_{di} = -c_i\,[(\dot{x} + \omega y)\vec{i} + (\dot{y} + \omega x)\vec{j}] \tag{9.30}$$

where c_i is the internal or rotary damping coefficient:

$$\text{External damping force: } \vec{F}_{de} = -c(\dot{x}\vec{i} + \dot{y}\vec{j}) \tag{9.31}$$

where C is the external damping coefficient. By substituting Eqs. (9.28) to (9.31) into Eq. (9.25), we obtain the equations of motion in scalar form:

$$m\ddot{x} + (c_i + c)\dot{x} + kx - c_i\omega y = m\omega^2 a\cos\omega t \tag{9.32}$$

$$m\ddot{y} + (c_i + c)\dot{y} + ky - c_i\omega x = m\omega^2 a\sin\omega t \tag{9.33}$$

These equations of motion, which describe the lateral vibration of the rotor, are coupled and are dependent on the speed of the steady-state rotation of the shaft, ω. By defining a complex quantity w as

$$w = x + iy \tag{9.34}$$

where $i = (-1)^{1/2}$, and by adding Eq. (9.32) to Eq. (9.33) multiplied by i, we obtain a single equation of motion:

$$m\ddot{w} + (c_i + c)\dot{w} + kw - i\omega c_i w = m\omega^2 a e^{i\omega t} \tag{9.35}$$

9.5.2 Critical Speeds

A critical speed is said to exist when the frequency of the rotation of a shaft equals one of the natural frequencies of the shaft. The undamped natural frequency of the rotor system can be obtained by solving Eqs. (9.32), (9.33), or (9.35), retaining only the homogeneous part with $c_i = c = 0$. This gives the natural frequency of the system (or critical speed of the undamped system):

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{9.36}$$

When the rotational speed is equal to this critical speed, the rotor undergoes large deflections, and the force transmitted to the bearings can cause bearing failures. A rapid transition of the rotating shaft through a critical speed is expected to limit the whirl amplitudes, while a slow transition through the critical speed aids the development of large amplitudes. Reference [9.15] investigates the behavior of the rotor during acceleration and deceleration through critical speeds. A FORTRAN computer program for calculating the critical speeds of rotating shafts is given in reference [9.16].